Nonlinear Analysis 152 (2017) 149–182

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Sobolev–Lorentz spaces in the Euclidean setting and counterexamples

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ARTICLE INFO

Article history:
Received 30 May 2016
Accepted 1 January 2017
Communicated by Enzo Mitidieri

MSC:
primary 31C45
46E35

Keywords:
Sobolev spaces
Lorentz spaces
Morrey embedding theorems

ABSTRACT

This paper studies the inclusions between different Sobolev–Lorentz spaces $W^{1,(p,q)}(\Omega)$ defined on open sets $\Omega \subset \mathbf{R}^n$, where $n \geq 1$ is an integer, $1 < p < \infty$ and $1 \leq q \leq \infty$. We prove that if $1 \leq q < r \leq \infty$, then $W^{1,(p,q)}(\Omega)$ is strictly included in $W^{1,(p,r)}(\Omega)$.

We show that although $H^{1,(p,\infty)}(\Omega) \subsetneq W^{1,(p,\infty)}(\Omega)$ where $\Omega \subset \mathbf{R}^n$ is open and $n \geq 1$, there exists a partial converse. Namely, we show that if a function u in $W^{1,(p,\infty)}(\Omega), n \geq 1$ is such that u and its distributional gradient ∇u have absolutely continuous (p,∞)-norm, then u belongs to $H^{1,(p,\infty)}(\Omega)$ as well.

We also extend the Morrey embedding theorem to the Sobolev–Lorentz spaces $H_0^{1,(p,q)}(\Omega)$ with $1 \leq n < p < \infty$ and $1 \leq q \leq \infty$. Namely, we prove that the Sobolev–Lorentz spaces $H_0^{1,(p,q)}(\Omega)$ embed into the space of Hölder continuous functions on $\overline{\Omega}$ with exponent $1 - \frac{n}{p}$ whenever $\Omega \subset \mathbf{R}^n$ is open, $1 \leq n < p < \infty$, and $1 \leq q \leq \infty$.

1. Introduction

In this paper we study the Sobolev–Lorentz spaces in the Euclidean setting and the inclusions between them. This paper is motivated by the results obtained in my 2006 Ph.D. thesis [6] and in my book [9]. There I studied the Sobolev–Lorentz spaces and the associated Sobolev–Lorentz capacities in the Euclidean setting for $n \geq 2$. The restriction on n there was due to the fact that I studied the n,q-capacity for $n > 1$.

The Sobolev–Lorentz spaces have also been studied by Cianchi–Pick in [4,5], by Kauhanen–Koskela–Malý in [22], and by Malý–Swanson–Ziemer in [25].

The classical Sobolev spaces were studied by Gilbarg–Trudinger in [15], Maz'ya in [26], Evans in [12], Heinonen–Kilpeläinen–Martio in [19], and by Ziemer in [30].

The Lorentz spaces were studied by Bennett–Sharpley in [1], Hunt in [21], and by Stein–Weiss in [29].

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http://dx.doi.org/10.1016/j.na.2017.01.001
0362-546X/

The Newtonian Sobolev spaces in the metric setting were studied by Shanmugalingam in [27,28]. See also Heinonen [18]. Costea–Miranda studied the Newtonian Lorentz Sobolev spaces and the corresponding global p,q-capacities in [11].

There are several other definitions of Sobolev-type spaces in the metric setting when $p = q$; see Hajłasz [16,17], Heinonen–Koskela [20], Cheeger [3], and Franchi–Hajłasz–Koskela [14]. It has been shown that under reasonable hypotheses, the majority of these definitions yields the same space; see Franchi–Hajłasz–Koskela [14] and Shanmugalingam [27].

The Sobolev–Lorentz relative p,q-capacity was studied in the Euclidean setting by Costea (see [6,7,9]) and by Costea–Maz'ya [10]. The Sobolev p-capacity was studied by Maz'ya [26] and by Heinonen–Kilpeläinen–Martio [19] in $\mathbf{R}^n$ and by J. Björn [2], Costea [8] and Kinnunen–Martio [23,24] in metric spaces.

The Sobolev–Lorentz spaces can be also studied in the Euclidean setting for $n = 1$. We do it in this paper. Many of the results on Sobolev–Lorentz spaces that we obtained in [6,9] in dimension $n \geq 2$ were extended here to the case $n = 1$.

In Section 3 we start by presenting some of the basic properties of the Lorentz spaces $L^{p,q}(\Omega; \mathbf{R}^m)$, where $\Omega \subset \mathbf{R}^n$ is open, $n, m \geq 1$ are integers, $1 < p < \infty$ and $1 \leq q \leq \infty$.

It is known that $L^{p,q}((0, \Omega_n r^n)) \subsetneq L^{p,s}((0, \Omega_n r^n))$. We see this in Theorem 3.4 by constructing a function u in $L^{p,s}((0, \Omega_n r^n)) \setminus L^{p,q}((0, \Omega_n r^n))$. Here $r > 0,\ n \geq 1,\ 1 < p < \infty$ and $1 \leq q < s \leq \infty$.

This function u is used in Theorem 3.5 to construct a radial function v that is smooth in the punctured ball $B^*(0,r)$ such that $|\nabla v|$ is in $L^{p,s}(B(0,r)) \setminus L^{p,q}(B(0,r))$. Later it will be shown in Theorem 4.13 that v is in $W^{1,(p,s)}(B(0,r)) \setminus W^{1,(p,q)}(B(0,r))$. This shows that the inclusion $W^{1,(p,q)}(B(0,r)) \subset W^{1,(p,s)}(B(0,r))$ is strict whenever $r > 0,\ n \geq 1,\ 1 < p < \infty$ and $1 \leq q < s \leq \infty$.

In Section 4 we revisit many of the results from my Ph.D. thesis [6, Chapter V] and from my book [9, Chapter 3] and we extend them to the case $n = 1$. We improve some of the old results from [6, Chapter V] and from [9, Chapter 3].

We also obtain some new results in this section. Among them we mention the case $q = \infty$ for Theorems 4.11 and 4.12 (see the discussion below) as well as the strict inclusion $W^{1,(p,q)}(B(0,r)) \subsetneq W^{1,(p,s)}(B(0,r))$ that we discussed above. As before, $r > 0,\ n \geq 1,\ 1 < p < \infty$ and $1 \leq q < s \leq \infty$ (see Theorem 4.13).

For $n \geq 2$, we proved in Costea [6,9] (by using partition of unity and convolution) that $H^{1,(p,q)}(\Omega) = W^{1,(p,q)}(\Omega)$ whenever $1 < p < \infty$ and $1 \leq q < \infty$. The partition of unity and convolution technique used there is similar to the techniques used by Ziemer in [30] and by Heinonen–Kilpeläinen–Martio in [19].

We proved in [6,9] (for $n \geq 2$) that $H^{1,(p,\infty)}(\Omega) \subsetneq W^{1,(p,\infty)}(\Omega)$. Once we constructed a function $u \in W^{1,(p,\infty)}(\Omega)$ such that its distributional gradient ∇u did not have an absolutely continuous (p,∞)-norm, we proved there that u was not in $H^{1,(p,\infty)}(\Omega)$.

In Section 4 of this paper, Proposition 4.7 and Theorem 4.8 show that $H^{1,(p,\infty)}(\Omega) \subsetneq W^{1,(p,\infty)}(\Omega)$ for $n \geq 1$. In this paper we also give a partial converse. Namely, we show in Theorem 4.11 that if a function u in $W^{1,(p,q)}(\Omega), n \geq 1,\ 1 \leq q \leq \infty$ is such that u and its distributional gradient ∇u have absolutely continuous (p,q)-norm, then u belongs to $H^{1,(p,q)}(\Omega)$ as well. This result is new for $q = \infty$ and $n \geq 1$ and improves a result from [6,9], proved there for $n \geq 2$ and $1 \leq q < \infty$. We proved this result via a partition of unity and convolution argument, because convolution and partition of unity work well on functions u that have absolutely continuous (p,q)-norm along with their distributional gradients ∇u.

In Theorem 4.12 we show that if a function u in $W^{1,(p,q)}(\mathbf{R}^n), n \geq 1$ is such that u and its distributional gradient ∇u have absolutely continuous (p,q)-norm, $1 \leq q \leq \infty$, then u belongs to $H_0^{1,(p,q)}(\mathbf{R}^n)$ as well. This result is new when $q = \infty$ and $n \geq 1$ and improves a result from [6,9], proved there for $n \geq 2$ and $1 \leq q < \infty$.

In Section 5 (among other things) we prove the Morrey embedding theorem for the Sobolev–Lorentz spaces $H_0^{1,(p,q)}(\Omega)$.