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Global regularity for supercritical nonlinear dissipative wave equations in 3D

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ABSTRACT

The nonlinear wave equation $u_{tt} - \Delta u + |u_t|^{p-1}u_t = 0$ is shown to be globally well-posed in the Sobolev spaces of radially symmetric functions $H^k_{\mathrm{rad}}(\mathbb{R}^3) \times H^{k-1}_{\mathrm{rad}}(\mathbb{R}^3)$ for all $p \geq 3$ and $k \geq 3$. Moreover, global C^∞ solutions are obtained when the initial data are C^∞_0 and exponent p is an odd integer.

The radial symmetry allows a reduction to the one-dimensional case where an important observation of Haraux (2009) can be applied, i.e., dissipative nonlinear wave equations contract initial data in $W^{k,q}(\mathbb{R}) \times W^{k-1,q}(\mathbb{R})$ for all $k \in [1, 2]$ and $q \in [1, \infty]$.

1. Introduction

Dissipative nonlinear wave equations are well-posed in $H^k(\mathbb{R}^n) \times H^{k-1}(\mathbb{R}^n)$ for all $k \in [1, 2]$and $n \geq 1$ under the monotonicity condition of Lions and Strauss [14]. This global result makes no exception for nonlinear dissipations of supercritical power, as determined by invariant scaling, and gets around the stringent conditions for well-posedness of general nonlinear wave equations; see Ponce and Sideris [17], Lindblad [13], Wang and Fang [26] and the references therein.

The monotonicity method has been less effective in studying higher regularity. It is still an open question whether supercritical problems are globally well-posed in Sobolev spaces with index $k > 2$. The purpose of this paper is to give an affirmative answer when $n = 3$ and initial data have radial symmetry. To state the result, let $\Box u = u_{tt} - \Delta u$ be the d'Alembertian in $\mathbb{R}^{3+1}$ and $Du = (\nabla u, \partial_t u)$ be the space-time gradient of u.

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We use $\|\cdot\|_q$, for the norm in $L^q(\mathbb{R}^3)$ with $q \in [1,\infty]$, and D^α, for the partial derivative of integer order $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$.

We consider the dissipative nonlinear wave equation

$$\Box u + |u_t|^{p-1} u_t = 0, \quad x \in \mathbb{R}^3,\ t > 0, \tag{1.1}$$

with the Cauchy data

$$u|_{t=0} = u_0, \qquad u_t|_{t=0} = u_1, \quad x \in \mathbb{R}^3. \tag{1.2}$$

Our main assumptions are $p \geq 3$ and $(u_0, u_1) \in H^k_{\mathrm{rad}}(\mathbb{R}^3) \times H^{k-1}_{\mathrm{rad}}(\mathbb{R}^3)$, where the radially symmetric Sobolev spaces are defined as

$$H^k_{\mathrm{rad}}(\mathbb{R}^3) = \{u \in H^k(\mathbb{R}^3) : (x_i \partial_{x_j} - x_j \partial_{x_i}) u(x) = 0 \text{ for } 1 \leq i < j \leq 3\}.$$

Clearly, such $u(x)$ depend only on $|x| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$. The radially symmetric spaces are invariant under the evolution determined by (1.1), (1.2); see [22,20].

Theorem 1.1. *Assume that $p \geq 3$ and $(u_0, u_1) \in H^3_{\mathrm{rad}}(\mathbb{R}^3) \times H^2_{\mathrm{rad}}(\mathbb{R}^3)$. Then problem (1.1), (1.2) admits a unique global solution u, such that*

$$D^\alpha u \in C([0,\infty), H^{3-|\alpha|}_{\mathrm{rad}}(\mathbb{R}^3)), \quad |\alpha| \leq 3.$$

Corollary 1.2. *The global solution of problem (1.1), (1.2), given by Theorem 1.1, satisfies the following uniform estimates: for $t \geq 0$,*

$$\sum_{1 \leq |\alpha| \leq 2} \int_0^t \|D^\alpha u(s)\|_\infty^2 ds \leq C_1(u_0, u_1),$$
$$\sum_{|\alpha| \leq 2} \|D D^\alpha u(t)\|_2 \leq C_2(u_0, u_1),$$

where $C_j(u_0, u_1),\ j = 1, 2$, are finite whenever $\|u_0\|_{H^3} + \|u_1\|_{H^2}$ is finite.

It is easy to derive the propagation of higher regularity from Theorem 1.1 and Sobolev embedding inequalities, if the nonlinearity allows further differentiation.

Theorem 1.3. *Let p be an odd integer and $(u_0, u_1) \in H^k_{\mathrm{rad}}(\mathbb{R}^3) \times H^{k-1}_{\mathrm{rad}}(\mathbb{R}^3)$ with an integer $k \geq 4$. Problem (1.1), (1.2) admits a unique global solution u, such that*

$$D^\alpha u \in C([0,\infty), H^{k-|\alpha|}_{\mathrm{rad}}(\mathbb{R}^3)), \quad |\alpha| \leq k.$$

In addition, the following estimates hold uniformly in $t \geq 0$:

$$\sum_{1 \leq |\alpha| \leq k-1} \int_0^t \|D^\alpha u(s)\|_\infty^2 ds \leq C_1^{(k)}(u_0, u_1),$$
$$\sum_{|\alpha| \leq k-1} \|D D^\alpha u(t)\|_2 \leq C_2^{(k)}(u_0, u_1),$$

where $C_j^{(k)}(u_0, u_1),\ j = 1, 2$, are finite whenever $\|u_0\|_{H^k} + \|u_1\|_{H^{k-1}}$ is finite.

If $(u_0, u_1) \in C^\infty_{\mathrm{rad}}(\mathbb{R}^3) \times C^\infty_{\mathrm{rad}}(\mathbb{R}^3)$ and (u_0, u_1) are compactly supported, then problem (1.1), (1.2) admits a unique global solution

$$u \in C^\infty([0,\infty), C^\infty_{\mathrm{rad}}(\mathbb{R}^3)),$$

such that $u(\cdot, t)$ is also compactly supported for all $t \geq 0$.