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Scattering theory for the defocusing energy-supercritical nonlinear wave equation with a convolution

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ABSTRACT

In this paper, we study the global well-posedness and scattering problem for the nonlinear wave equation with a convolution $u_{tt} - \Delta u + (|x|^{-\gamma} * |u|^2)u = 0$ in dimensions $d \geq 6$. We show that if the solution u is apriorily bounded in the critical homogeneous Sobolev space, that is, $u \in L_t^\infty(\dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1})$, with $s_c := \frac{\gamma-2}{2} > 1$, then u is global and it scatters. The impetus to consider this problem stems from a series of recent works for the energy-supercritical nonlinear wave equation and nonlinear Schrödinger equation. Our analysis derived from the concentration compactness method to show that the proof of the global well-posedness and scattering is reduced to disprove the existence of three scenarios: the finite time blow-up solution, the soliton-like solution and the low-to-high frequency cascade. We note that, authors preclude the finite time blow-up solution to wave equation usually by the property of the finite speed of propagation in the previous literature, however, the finite speed of propagation is broken in the nonlocal nonlinear wave equation with a convolution nonlinearity. For this, we will initially establish the low regularity results for almost periodic solutions including the finite time blow-up solutions and initially preclude the finite time blow-up solutions by lowering the regularity.

1. Introduction

In this paper, we shall consider the initial value problem for the defocusing nonlinear wave equation with a Convolution in dimensions $d \geq 6$:

$$\text{(NLWH)} \quad \begin{cases} u_{tt} - \Delta u + (|x|^{-\gamma} * |u|^2)u = 0, \quad d \geq 6, \\ (u, u_t)|_{t=0} = (u_0, u_1) \in \dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}(\mathbb{R}^d), \end{cases}$$

where the nonlinearity $F(u) = (|x|^{-\gamma} * |u|^2)u$ is energy-supercritical, that is, $4 < \gamma < d$; and $u(t,x)$ is a real-valued function on $I \times \mathbb{R}^d$ and $0 \in I \subset \mathbb{R}$ is a time interval, $s_c = \frac{\gamma-2}{2}$.

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Formally, the solution u of (NLWH) conserves the energy

$$E(u(t), u_t(t)) = \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla u(t)|^2 + \frac{1}{2}|u_t(t)|^2\right) dx + \frac{1}{4}\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u(t,x)|^2|u(t,y)|^2}{|x-y|^\gamma} dxdy$$
$$\equiv E(u_0, u_1).$$

The class of solutions to wave equation $u_{tt} - \Delta u + (|x|^{-\gamma} * |u|^2)u = 0$ is left invariant by the scaling

$$u(t,x) \mapsto u_\lambda(t,x) = \lambda^{\frac{2+d-\gamma}{2}} u(\lambda t, \lambda x) \quad \forall \lambda > 0.$$

Moreover, it leaves the Sobolev norm $\dot{H}^{s_c}_x(\mathbb{R}^d)$ with $s_c = \frac{\gamma-2}{2}$ invariant, this defines a notion of criticality. In view of the cubic convolution nonlinear term, $4 < \gamma < d$ implies $s_c > 1$, and therefore we say that $4 < \gamma < d$ is the *energy-supercritical* regime for (NLWH). We study (NLWH) in the energy-supercritical regime $s_c > 1$, in dimensions $d \geq 6$ in this paper. We consider *solutions* to (NLWH), that is, functions $u : I \times \mathbb{R}^d \to \mathbb{R}$ such that for every $K \subset I$ compact, $(u, u_t) \in C_t(K; \dot{H}^{s_c}_x \times \dot{H}^{s_c-1}_x)$, $u \in L^{\frac{2(1+d)}{d+2-\gamma}}_{t,x}(K \times \mathbb{R}^d)$, and satisfying the *Duhamel formula*

$$u(t) = \mathcal{W}(t)(u_0, u_1) + \int_0^t \frac{\sin((t-t')|\nabla|)}{|\nabla|} F(u(t'))dt'$$

for every $t \in I$, where $0 \in I \subset \mathbb{R}$ is a time interval and the wave propagator

$$\mathcal{W}(t)(u_0, u_1) = \cos(t|\nabla|)u_0 + \frac{\sin(t|\nabla|)}{|\nabla|} u_1$$

is the solution to the linear wave equation with initial data (u_0, u_1).

We refer to I as the *interval of existence* of u, and we say that I is the *maximal interval of existence* if u cannot be extended to any larger time interval. We say that u is a *global solution* if $I = \mathbb{R}$, and that u is a *blow-up solution* if $\|u\|_{L^{\frac{2(1+d)}{d+2-\gamma}}_{t,x}(I\times\mathbb{R}^d)} = \infty$.

Now we state our main result:

Theorem 1.1. *Assume that $d \geq 6$, $s_c = \frac{\gamma-2}{2}$ and $\frac{4d}{d-1} < \gamma < d$. Let $u : I \times \mathbb{R}^d \to \mathbb{R}$ be a solution to (NLWH) with maximal interval of existence $I \subset \mathbb{R}$ satisfying*

$$(u, u_t) \in L^\infty_t(I; \dot{H}^{s_c}_x \times \dot{H}^{s_c-1}_x). \tag{1.1}$$

Then u is global and

$$\int_{\mathbb{R}}\int_{\mathbb{R}^d} |u(t,x)|^{\frac{2(1+d)}{d+2-\gamma}} dxdt \leq C,$$

for some constant $C = C(\|(u,u_t)\|_{L^\infty_t(I;\dot{H}^{s_c}_x\times\dot{H}^{s_c-1}_x)})$.

Moreover, u scatters in the sense that there exist unique $(u_0^\pm, u_1^\pm) \in \dot{H}^{s_c}_x \times \dot{H}^{s_c-1}_x$ such that

$$\lim_{t\to\pm\infty} \|(u(t), u_t(t)) - (\mathcal{W}(t)(u_0^\pm, u_1^\pm), \partial_t \mathcal{W}(t)(u_0^\pm, u_1^\pm))\|_{\dot{H}^{s_c}_x\times\dot{H}^{s_c-1}_x} = 0.$$

Remark 1.1. The restriction $\frac{4d}{d-1} < \gamma$ in Theorem 1.1 stems from Lemmas 4.2 and 4.3. More precisely, we need to find an exponent $q = q(d)$ simultaneously satisfies that $q(d) \in (\frac{2(d-1)}{d-3}, \frac{2d}{d+2-\gamma})$ and $\frac{4d}{3d-2\gamma} < q(d) < \frac{2d(d-1)}{d^2+d-\gamma d+\gamma}$, when $d \geq 6$. These conditions on $q(d)$ impose the restriction $\frac{4d}{d-1} < \gamma < d$. One may refer to Section 4 for more details.

The highlight of the paper is that we establish the low regularity results Theorem 4.1 for almost periodic solutions including the finite time blow-up solutions. In Section 5, we preclude the finite time blow-up solution by the low regularity results, while the previous papers in the literature, they use the finite speed