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Twist maps as energy minimisers in homotopy classes: Symmetrisation and the coarea formula

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A B S T R A C T

Let $\mathbb{X} = \mathbb{X}[a, b] = \{x : a < |x| < b\} \subset \mathbb{R}^n$ with $0 < a < b < \infty$ fixed be an open annulus and consider the energy functional,

$$\mathbb{F}[u; \mathbb{X}] = \frac{1}{2} \int_{\mathbb{X}} \frac{|\nabla u|^2}{|u|^2} \, dx,$$

over the space of admissible incompressible Sobolev maps

$$\mathcal{A}_\phi(\mathbb{X}) = \left\{ u \in W^{1,2}(\mathbb{X}, \mathbb{R}^n) : \det \nabla u = 1 \ a.e. \text{ in } \mathbb{X} \text{ and } u|_{\partial\mathbb{X}} = \phi \right\},$$

where ϕ is the identity map of $\overline{\mathbb{X}}$. Motivated by the earlier works (Taheri (2005), (2009)) in this paper we examine the *twist* maps as extremisers of $\mathbb{F}$ over $\mathcal{A}_\phi(\mathbb{X})$ and investigate their minimality properties by invoking the coarea formula and a symmetrisation argument. In the case $n = 2$ where $\mathcal{A}_\phi(\mathbb{X})$ is a union of infinitely many disjoint homotopy classes we establish the minimality of these extremising twists in their respective homotopy classes a result that then leads to the latter twists being L^1-local minimisers of $\mathbb{F}$ in $\mathcal{A}_\phi(\mathbb{X})$. We discuss variants and extensions to higher dimensions as well as to related energy functionals.

1. Introduction and preliminaries

Let $\mathbb{X} = \mathbb{X}[a, b] = \{(x_1, \dots, x_n) : a < |x| < b\}$ with $0 < a < b < \infty$ fixed be an open annulus in $\mathbb{R}^n$ and consider the energy functional

$$\mathbb{F}[u; \mathbb{X}] = \frac{1}{2} \int_{\mathbb{X}} \frac{|\nabla u|^2}{|u|^2} \, dx, \tag{1.1}$$

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over the space of incompressible Sobolev maps,

$$\mathcal{A}_\phi(\mathbb{X}) = \left\{ u \in W^{1,2}(\mathbb{X}, \mathbb{R}^n) : \det \nabla u = 1 \ a.e. \text{ in } \mathbb{X} \text{ and } u|_{\partial \mathbb{X}} = \phi \right\}. \tag{1.2}$$

Here and in future ϕ denotes the identity map of $\overline{\mathbb{X}}$ and so the last condition in (1.2) means that $u \equiv x$ on $\partial \mathbb{X}$ in the sense of traces.

By a twist map u on $\mathbb{X} \subset \mathbb{R}^n$ we mean a continuous self-map of $\overline{\mathbb{X}}$ onto itself which agrees with the identity map ϕ on the boundary $\partial \mathbb{X}$ and has the specific spherical polar representation (*see* [16–18] for background and further results)

$$u : (r, \theta) \mapsto (r, Q(r)\theta), \quad x \in \overline{\mathbb{X}}. \tag{1.3}$$

Here $r = |x|$ lies in $[a, b]$ and $\theta = x/|x|$ sits on $\mathbb{S}^{n-1}$ with $Q \in \mathbf{C}([a, b], \mathbf{SO}(n))$ satisfying $Q(a) = Q(b) = I$. Therefore Q forms a closed loop in $\mathbf{SO}(n)$ based at I and for this in sequel we refer to Q as the twist *loop* associated with u. Also note that (1.3) in Cartesian form can be written as

$$u : x \mapsto Q(r)x = rQ(r)\theta, \quad x \in \overline{\mathbb{X}}. \tag{1.4}$$

Next subject to a differentiability assumption on the twist loop Q it can be verified that $u \in \mathcal{A}_\phi(\mathbb{X})$ with its $\mathbb{F}$ energy simplifying to

$$\begin{aligned} \mathbb{F}[Q(r)x; \mathbb{X}] &= \frac{1}{2} \int_{\mathbb{X}} \frac{|\nabla u|^2}{|u|^2} \, dx = \frac{1}{2} \int_{\mathbb{X}} \frac{|\nabla Q(r)x|^2}{|x|^2} \, dx \\ &= \frac{n}{2} \int_{\mathbb{X}} \frac{dx}{|x|^2} + \frac{\omega_n}{2} \int_a^b |\dot{Q}|^2 r^{n-1} \, dr, \end{aligned} \tag{1.5}$$

where the last equality uses $|\nabla[Q(r)x]|^2 = n + r^2|\dot{Q}\theta|^2$. Now as the primary task here is to search for extremising twist maps we first look at the Euler–Lagrange equation associated with the loop energy $\mathbb{E} = \mathbb{E}[Q]$ defined by the last integral in (1.5) over the loop space $\{Q \in W^{1,2}([a, b]; \mathbf{SO}(n)) : Q(a) = Q(b) = I\}$. Indeed this can be shown to take the form (see below for justification)

$$\frac{d}{dr} \left[\left(r^{n-1} \dot{Q} \right) Q^t \right] = 0, \tag{1.6}$$

with solutions $Q(r) = \exp[-\beta(r)A]P$, where $P \in \mathbf{SO}(n)$, $A \in \mathbb{R}^{n \times n}$ is skew-symmetric and $\beta = \beta(|x|)$ is described for $a \le r \le b$ by

$$\beta(r) = \begin{cases} \ln 1/r & n = 2, \\ r^{2-n}/(n-2) & n \ge 3. \end{cases} \tag{1.7}$$

Now to justify (1.6) fix $Q \in W^{1,2}([a, b], \mathbf{SO}(n))$ and for $F \in W_0^{1,2}([a, b], \mathbb{R}^{n \times n})$ set $H = (F - F^t)Q$ and $Q_\epsilon = Q + \epsilon H$. Then $Q_\epsilon^t Q_\epsilon = I + \epsilon^2 H^t H$ and

$$\begin{aligned} \frac{d}{d\epsilon} \int_a^b 2^{-1} |\dot{Q}_\epsilon|^2 r^{n-1} \, dr \Big|_{\epsilon=0} &= \int_a^b \langle \dot{Q}, (\dot{F} - \dot{F}^t)Q + (F - F^t)\dot{Q} \rangle r^{n-1} \, dr \\ &= \int_a^b \langle \dot{Q}, (\dot{F} - \dot{F}^t)Q \rangle r^{n-1} \, dr \\ &= \int_a^b \left\langle \frac{d}{dr}(r^{n-1} \dot{Q} Q^t), (F - F^t) \right\rangle dr = 0, \end{aligned}$$

and so the arbitrariness of F with an orthogonality argument gives (1.6).

Returning to (1.1) it is not difficult to see that the Euler–Lagrange equation associated with $\mathbb{F}$ over $\mathcal{A}_\phi(\mathbb{X})$ is given by the system (*cf.* Section 4)

$$\frac{|\nabla u|^2}{|u|^4} u + div \left\{ \frac{\nabla u}{|u|^2} - p(x) \text{cof} \, \nabla u \right\} = 0, \quad u = (u_1, \ldots, u_n), \tag{1.8}$$