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On a function projective synchronization scheme for non-identical Fractional-order chaotic (hyperchaotic) systems with different dimensions and orders

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ABSTRACT

Referring to fractional-order systems, this paper investigates the *inverse full state hybrid function projective synchronization* (IFSHFPS) of *non-identical* systems characterized *by different dimensions* and *different orders*. By taking a master system of dimension n and a slave system of dimension m, the method enables each master system state to be synchronized with a linear combination of slave system states, where the scaling factor of the linear combination can be any arbitrary differentiable function. The approach presents some useful features: i) it enables commensurate and incommensurate non-identical fractional-order systems with different dimension $n<m$ or $n>m$ to be synchronized; ii) it can be applied to a wide class of chaotic (hyperchaotic) fractional-order systems for any differentiable scaling function; iii) it is rigorous, being based on two theorems, one for the case $n<m$ and the other for the case $n>m$. Two different numerical examples are reported, involving chaotic/hyperchaotic fractional-order Lorenz systems (three-dimensional and four-dimensional master/slave, respectively) and hyperchaotic/chaotic fractional-order Chen systems (four-dimensional and three-dimensional master/slave, respectively). The examples clearly highlight the capability of the conceived approach in effectively achieving synchronized dynamics for any differentiable scaling function.

1. Introduction

Chaos synchronization was discovered by Pecora and Carroll in dynamical systems described by *integer-order* differential equations [1]. The objective in chaos synchronization is to make the slave system variables synchronized in time with the corresponding chaotic master system variables [1]. Different types of synchronization have been proposed in the literature over the years [2–6]. Very recently, the *full state hybrid projective synchronization* (FSHPS) has been proposed, where each slave system state synchronize with a linear combination of master system states [5]. On the other hand, in [6] a synchronization scheme has been presented, where each master system state synchronizes with a linear combination of slave system states. Since master system states and slave system states have been inverted in [6] with respect to the FSHPS, the scheme in [6]

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has been called *inverse full-state hybrid projective synchronization* (IFSHPS). Additionally, when the scaling factor is replaced by a scaling function, the *inverse full-state hybrid function projective synchronization* (IFSHFPS) is obtained [7].

Besides integer-order systems, attention has been recently focused on dynamical systems described by *fractional-order* differential equations, i.e., systems where the order of the derivative is a non-integer number [8]. The conducted studies have shown that fractional-order systems are characterized by chaotic dynamics [8–10]. To this purpose, it has been shown that in fractional-order systems chaos is achievable when the system order is less than 3, whereas hyperchaos can be obtained when the system order is less than 4 [8]. Referring to synchronization, it is worth noting that, differently from integer-order systems, few synchronization types have been introduced for fractional-order systems to date. Moreover, most of the approaches are related to the synchronization of *identical* fractional-order systems [10]. An interesting approach for synchronizing *non-identical* fractional-order chaotic systems has been illustrated in [11]. On the other hand, referring to *function projective* synchronization, only few scheme have been proposed [10–13].

In this paper a further contribution to the topic is provided. Namely, the manuscript investigates the *inverse full state hybrid function projective synchronization* (IFSHFPS) of *non-identical* fractional-order systems characterized *by different dimensions* and *different orders*. Specifically, each master system state synchronizes with a linear combination of slave system states, where *the scaling factor of the linear combination can be any arbitrary differentiable function*. The method can be applied either when the dimension n of the master system is less than the dimension m of the slave system or vice versa. The approach presents some useful features: i) it can be applied to a wide class of chaotic (hyperchaotic) fractional-order systems; ii) it enables commensurate and incommensurate fractional-order systems with different dimension $n<m$ or $n>m$ to be synchronized; iii) the scaling factor of the linear combination can be any arbitrary differentiable function; and iv) it is rigorous, being based on two theorems, one for the case $n<m$ and the other for the case $n>m$.

The paper is organized as follows. In Section 2, the basic notions on fractional calculus are given. In Section 3, the IFSFHPS for fractional-order systems is introduced. The conceived scheme is general and the only restriction on the scaling functions is that they must be differentiable functions. In Section 4, by exploiting classical Lyapunov stability theory, the IFSHFPS between two fractional-order systems for the case $n<m$ is proved, showing that the zero solution of the error system is globally asymptotically stable. Additionally, by using the stability theory of linear integer-order systems, the IFSHFPS between two fractional-order systems for the case $n>m$ is proved. In Section 5, different numerical examples are provided. At first the IFSHFPS between a three-dimensional chaotic fractional-order commensurate Lorenz system and a four-dimensional hyperchaotic fractional-order incommensurate Lorenz system is successfully achieved. Note that the scaling functions include exponential or polynomial functions. Successively, a numerical example illustrates the IFSHFPS between a four-dimensional hyperchaotic fractional-order commensurate Chen system and a three-dimensional chaotic fractional-order commensurate Chen system. By plotting the error dynamics, it can be readily shown the effectiveness of the proposed theoretical approach in successfully achieving IFSHFPS between commensurate and incommensurate fractional-order systems of different dimensions, for any differentiable scaling function.

2. Basic notions on fractional calculus

The Riemann-Liouville fractional integral operator of order $q>0$ of the function $f(t)$ is defined as:

$$J^q f(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-\tau)^{q-1} f(\tau)\, d\tau,\ t>0. \tag{1}$$

where Γ denotes Gamma function [14,15].

Herein, the Caputo definition is used, according to which the fractional derivative of f(t) is [16]:

$$D_t^p f(t) = J^{m-p}\left(\frac{d^m}{dt^m} f(t)\right) = \frac{1}{\Gamma(m-p)} \int_{t_0}^{t} \frac{f^{(m)}(\tau)}{(t-\tau)^{p-m+1}}\, d\tau, \tag{2}$$

for $m-1<p\le m$, $m \in \mathbb{N}$ and $t>0$.

Now some useful lemmas are reported.

Lemma 1. [17] *The Laplace transform of the Caputo fractional derivative (2) is:*

$$\mathbf{L}\left(D_t^p f(t)\right) = s^p \mathbf{F}(s) - \sum_{k=0}^{n-1} s^{p-k-1} f^{(k)}(0),\ (p>0,\ n-1<p\le n). \tag{3}$$

In particular, when $0<p\le 1$, it results:

$$\mathbf{L}\left(D_t^p f(t)\right) = s^p \mathbf{F}(s) - s^{p-1} f(0). \tag{4}$$

Lemma 2. [18] *The Laplace transform of the Riemann-Liouville fractional integral (1) is:*