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Asymptotic Preserving scheme for strongly anisotropic parabolic equations for arbitrary anisotropy direction

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ABSTRACT

This paper deals with the numerical study of a strongly anisotropic heat equation. The use of standard schemes in this situation leads to poor results, due to high anisotropy. Furthermore, the recently proposed Asymptotic-Preserving method (Lozinski et al., 2012) allows one to perform simulations regardless of the anisotropy strength but its application is limited to the case where the anisotropy direction is given by a field whose lines are all open. In this paper we introduce a new Asymptotic-Preserving method, which overcomes those limitations without any loss of precision or increase in computational costs. The convergence of the method is shown to be independent of the anisotropy parameter $0 < \varepsilon < 1$ for fixed coarse Cartesian grids, and for variable anisotropy directions. The context of this work is magnetically confined fusion plasmas.

1. Introduction

This work deals with the efficient numerical treatment of heat transport in a strongly anisotropic medium. In particular, we address models of magnetized plasma with magnetic field perturbations, such as those produced by tearing modes and magnetic islands.

In classical transport theory of strongly magnetized plasmas, the ratio of the parallel ($\chi_\parallel$) to the perpendicular ($\chi_\perp$) heat conductivity of a given species (electrons or ions) scales like $(\Omega_c\tau_c)^2$, where Ω_c is the cyclotron frequency (the rotation frequency around the field lines) and τ_c the collision frequency. This product is very large; in the case of electrons, it can be of the order of 10^{10} in the weakly collisional plasmas typical of thermonuclear reactors such as ITER [1].

In plasma theory and simulations, one usually considers a reference (primary) magnetic field. It is usually, but not necessarily, the solution of a plasma equilibrium equation expressing the balance between electromagnetic and pressure forces.

Often, the primary magnetic field has a symmetry, such as the axisymmetry of a tokamak device [2]. In three dimensions, a symmetry implies the existence of magnetic surfaces on which field lines lie. Plasma transport across these surfaces depends on the perpendicular transport coefficients, and since they are small, good plasma confinement would be expected when these surfaces are closed. Conversely, transport on a magnetic surface, being controlled by parallel transport coefficients, is comparatively fast. As a result, plasma states are characterized by almost constant values of the various physical quantities (density, temperatures, etc.) on a given magnetic surface, with gradients directed mainly across such surfaces.

However, it turns out that in many applications the symmetry of the primary magnetic field configuration is broken either by dynamical effects, often the consequence of instabilities, or by external perturbations. As a result, closed magnetic surfaces can be broken, leading to increased transport.

This is the case with magnetic islands [3]. They can be seen as secondary magnetic structures occurring when a perturbation with a small component of the magnetic field pointing in the direction perpendicular to the original magnetic surfaces produces another set of magnetic surfaces with distinct topology.

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To be more precise, let us consider perhaps the simplest interesting case in two dimensions (2D), in some rectangular domain with Cartesian coordinates (x, y). Assume a primary field $B_y(x) = x$. The original magnetic surfaces (reduced to lines in this 2D case) are given by $x =$ const. Transport in the x direction is limited by the small perpendicular conductivity, whereas transport along y is fast. A quantity like the temperature T would settle very quickly into a state such that $T = T(x)$, some function of x. Add now a small perturbation with a component pointing along x, such that $B_x(y) = A\cos(y)$, with A a (usually small) constant. One can see that, regardless the value of A, the domain is split into three regions of which the one comprised within the curve $x^2 = 2A[1 + \sin(y)]$ (called the separatrix) has closed field lines. This region is called a "magnetic island". Outside the island the field lines are only bent, whereas inside they form closed loops which can be thought of as the result of reconnecting two former field lines initially on either side of the $x = 0$ line and with magnetic field vectors pointing in the opposite directions. Hence the name of "magnetic reconnection" given to this process.

Inside the island, and along each closed field line, a quantity like the temperature would be roughly constant, as a consequence of the strong parallel conductivity. In steady state one expects that the temperature would be roughly constant across the whole island, unless energy is fed inside the island. Thus islands act as an effective short circuit for radial transport and they are therefore often unwanted effects in confinement devices.

Theories of the formation of magnetic islands rely on various ingredients. In the regime where tearing modes (TM) [4] are linearly unstable, magnetic islands are the result of TM evolution [5] and saturation [6,7]. When, however, TMs are stable, magnetic islands can still occur through a mechanism of self-sustainment [8]. In this regime, a key element of the island dynamics is the competition between the parallel and the perpendicular heat fluxes, depending in particular on the ratio $\chi_\parallel/\chi_\perp$ ($\equiv 1/\varepsilon$), which may ultimately determine whether the island grows or is suppressed [9].

One can then see the interest of studying the heat (or diffusion) equation with very strong anisotropy, whether in the context of magnetic island dynamics or of other applications. This equation can occur in numerical simulations as a standalone problem, or as part of a more complex problem. In the latter case it would occur as a consequence of splitting the full problem in sub-problems for more effective coding.

For a given quantity u, representing for example density or temperature, the heat equation studied in this paper can be written as

$$\partial_t u - \frac{1}{\varepsilon}\nabla_\parallel \cdot (A_\parallel \nabla_\parallel u) - \nabla_\perp \cdot (A_\perp \nabla_\perp u) = 0, \tag{1}$$

where $\nabla_\parallel$ and $\nabla_\perp$ denote the gradient in the direction parallel (respectively perpendicular) to the magnetic field. Depending on boundary conditions, this problem can become ill-posed in the limit of $\varepsilon \to 0$.

Conventional numerical methods usually fail to give accurate results with reasonable computational resources for realistic physical parameters. Indeed, when $1/\varepsilon$ is of the order of 10^{10} the problem becomes extremely anisotropic and a standard discretization leads to very badly conditioned linear systems with condition number proportional to $1/(\varepsilon h^2)$ (h being the spatial discretization step). It is therefore important to develop a numerical scheme that can address the problem and give accurate results independently of the value of ε.

The original motivation of this work comes from fusion plasma physics, but similar anisotropic problems are encountered in many other fields. For example one can mention image processing [10,11], transport modelling in fractured geological structures [12] or semiconductor modelling [13].

Numerical resolution of strongly anisotropic problems has been addressed by many authors. For example, suitable coordinates are often used in the context of plasma simulations [14–16]. However, this approach can be difficult to implement, especially when the magnetic field is variable in time. This is why it is preferable to choose a method which does not require mesh or coordinate adaptation, like in [17]. Another approach relies on numerical schemes specially developed for anisotropic problems. Finite difference schemes were investigated in [18–20]. A high order finite element method was proposed in [21]. Multigrid methods [22] can sometimes be beneficial. These methods are usually efficient for a selected range of ε but do not behave well in the limit $\varepsilon \to 0$.

A different way to overcome this difficulty (adopted in this paper) is to apply the so called Asymptotic Preserving scheme introduced first in [23] to deal with singularly perturbed kinetic models. The idea is to reformulate the initial problem into an equivalent form, which remains well-posed in the limit of infinite anisotropy strength. A similar reformulation was applied to the anisotropic stationary diffusion equation in [24] and then to the nonlinear anisotropic heat equation in [25,26]. The method presented therein is based on the introduction of an auxiliary variable, which serves to eliminate from the equation the dominant part, *i.e.* the one multiplied by $1/\varepsilon$. The choice of the auxiliary variable presented in those papers allows one to solve the problem regardless of the anisotropy strength but imposes serious limitations on the magnetic field. In particular, the case of magnetic islands cannot be treated by those schemes.

In this paper we propose a new method which overcomes this limitation. The difference between this new scheme and the one presented in [26] lies in the choice of the auxiliary variable q. In the previous work q is defined by a parallel gradient of u: $\nabla_\parallel q = \frac{1}{\varepsilon}\nabla_\parallel u$. In order to ensure uniqueness, the value of q is set to 0 on the part of the boundary where field lines enter the domain. This clearly limits the application of the method only to the domains with open field lines. Indeed, if a closed line is present, then q is defined up to a function constant along this closed line. In order to overcome this drawback we propose an alternative approach. The auxiliary function still satisfies the parallel gradient relation. However this time it is required to be in the space of functions with zero average along all field lines. This requirement provides uniqueness of the auxiliary variable independently of the magnetic field topology and thus clearly resolves the limitations of the previous method. However, in the general case of arbitrary magnetic fields, this kind of space is far from being easy to discretize. This problem is overcome with a penalty stabilization technique well known in the context of Stokes equations. The idea is to relax the parallel gradient relation between u and q with a suitable stabilization term. This term ensures that the auxiliary variable has zero average along the field lines at the price of some (small) additional error, which does not alter the convergence rate.

The plan of the article is as follows. In Section 2 the mathematical problem is presented. Section 3 is devoted to the description of the numerical method. The numerical tests with known analytic solutions and the application to the problem of transport in a magnetic island are presented in Section 4.