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Hexagonal images processing over finite Eisenstein fields

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Abstract

This paper considers a new algebraic method for analysis and processing of hexagonally sampled images. The method is based on the interpretation of such images as functions on "Eisenstein fields". These are finite fields $\mathbb{GF}(p^2)$ of special characteristics $p = 12k + 5$, where $k > 0$ is an integer. Some properties of such fields are studied; in particular, it is shown that its elements may be considered as "discrete Eisenstein numbers" and are in natural correspondence with hexagons in a $(p \times p)$-diamond-shaped fragment of a regular plane tiling. We show that in some cases multiplications in Eisenstein fields correspond to rotations combined with appropriate scalings, and use this fact for hexagonal images sharpening, smoothering and segmentation. The proposed algorithms have complexity $O(p^2)$ and can be used also for processing of square-sampled digital images over finite Gaussian fields.

Keywords: hexagonal image; processing; Eisenstein numbers; finite fields; rotations; sharpening; smoothering; segmentation

1. Introduction

Hexagonal sampling (as in Fig. 1) is not new and has been explored by many researchers [1–5]. Indeed, it is known [1] that hexagonal lattice has some advantages over the square lattice which can have implications for analysis of images defined on it. These advantages are as follows:

- **Isoperimetry**. As per the honeycomb conjecture, a hexagon encloses more area than any other closed planar curve of equal perimeter, except a circle.
- **Additional equidistant neighbours**. Every hexagonal pixel has six equidistant neighbours with a shared edge. In contrast, a square pixel has only four equidistant neighbours with a shared edge or a corner. This implies that curves can be represented in a better fashion on the hexagonal lattice.
- **Additional symmetry axes**. Every hexagon in the lattice has 6 symmetry axes in contrast with squares, which have only 4 axes. This implies that there will be less ambiguity in detecting symmetry of images.

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Fig. 1. An example of hexagonal image.

In general, the hexagonal structure provides a more flexible and efficient way to perform image translation and rotation without losing image information [6] and demonstrate the ability to better represent curved structures [7].

A considerable amount of research in hexagonally sampled images processing is taking place now despite the fact that there are no hardware resources that currently produce or display hexagonal images. For this, software resampling is in use, when the original data is sampled on a square lattice while the desired image is to be sampled on a hexagonal lattice; in particular, it has been used to produce examples for the current paper. (Note that for the sake of brevity, the terms *square image* and *hexagonal image* will be used throughout to refer to images sampled on a square lattice and hexagonal lattice, respectively.)

When developing algorithms for image analysis, both in square and hexagonal grid, it is quite common to proceed on the assumption of continuity of images. Then powerful tools of continual mathematics, such as complex analysis and integral transforms, can be efficiently used. However, its application to digital images often leads to systematic errors associated with the inability to adequately transfer some concepts of continuous mathematics to the discrete plane. As an example we can point to the concept of rotation in the plane. Being natural and elementary in the continuous case, it lose these qualities when one tries to define it accurately on a discrete plane. As a result, formal application of continuous methods to digital images could be complicated by systematic errors. This raises the issue of the development of methods initially focused on discrete images and based on tools of algebra and number theory.

One of such methods is considered in this paper. It is based on the interpretation of hexagonal images as functions on “Eisenstein fields”. These are finite fields $\mathbb{GF}(p^2)$ of special characteristics $p = 12k+5$, where $0 \le k \in \mathbb{Z}$. We show that elements of such fields may be considered as “discrete Eisenstein numbers” and are in natural correspondence with hexagons in a $(p \times p)$-diamond-shaped fragment of a regular plane tiling. Hence, functions on Eisenstein fields may be considered as hexagonal images of sizes $(p \times p)$, where $p = 5, 17, \ldots, 257, \ldots, 509$, ets. (Note that it is not a limitation since every image can be trivially extended to an appropriate size.)

The significance of such approach is based on the fact that Eisenstein fields inherit some properties of the continuous complex field $\mathbb{C}$. In particular, it is well-known that in the complex plane multiplications may be considered as rotations combined with appropriate scalings. We show that in some respects it is true also for multiplication in Eisenstein fields. Namely, it occurs that though in general such multiplication do produce considerable distortion of images, in some special cases a resulting image consists of several fragments, visually similar to the original one rotated and zoomed out. We use the fragments after such *Eisenstein rotation* to produce masks for sharpening, smoothering and segmentation of images; the crucial point here is the invertibility of Eisenstein rotations. All the proposed algorithms have complexity $O(p^2)$ and can also be used for square-sampled digital images processing over *finite Gaussian fields* [8,9] or *complex discrete tori* [10–13], see also [14,15].