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Probabilistic solutions of some multi-degree-of-freedom nonlinear stochastic dynamical systems excited by filtered Gaussian white noise

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ABSTRACT

In this paper, the state-space-split method is extended for the dimension reduction of some high-dimensional Fokker–Planck–Kolmogorov equations or the nonlinear stochastic dynamical systems in high dimensions subject to external excitation which is the filtered Gaussian white noise governed by the second order stochastic differential equation. The selection of sub state variables and then the dimension-reduction procedure for a class of nonlinear stochastic dynamical systems is given when the external excitation is the filtered Gaussian white noise. The stretched Euler–Bernoulli beam with hinge support at two ends, point-spring supports, and excited by uniformly distributed load being filtered Gaussian white noise governed by the second-order stochastic differential equation is analyzed and numerical results are presented. The results obtained with the presented procedure are compared with those obtained with the Monte Carlo simulation and equivalent linearization method to show the effectiveness and advantage of the state-space-split method and exponential polynomial closure method in analyzing the stationary probabilistic solutions of the multi-degree-of-freedom nonlinear stochastic dynamical systems excited by filtered Gaussian white noise.

1. Introduction

Randomness always exists in the real world and hence many problems in science and engineering can be described with multi-degree-of-freedom (MDOF) nonlinear stochastic dynamical (NSD) systems. The probabilistic solutions of NSD systems excited by white noise or filtered white noise are governed by Fokker–Planck–Kolmogorov (FPK) equations. Hence the application of NSD systems and the formulation of FPK equations can be found in various areas of science and engineering [1–5]. However, it is difficult to obtain the exact solutions of the FPK equations formulated from real nonlinear problems. Only under some restrictive conditions, the stationary exact solutions are obtainable for some one, two, or few-dimensional systems. In order to solve real problems, some methods were introduced or extended for obtaining the approximate probabilistic solutions of NSD systems, such as the path integral method [6–8], stochastic average method [9], perturbation method [10–12], Gram–Charlier series or Hermite-polynomial closure method [13], finite element method [14], finite difference method [15], and exponential polynomial closure (EPC) method [16,17]. It is known that these methods were used for analyzing the one or two-degree-of-freedom systems or solving the FPK equations in one to four dimensions.

Because of the challenges in analyzing the probabilistic solutions of MDOF NSD systems, the problem of solving the FPK equations in high dimensions or obtaining the probabilistic solutions of NSD systems in high dimensions attracted a lot of research interests in the last one century. There are two methods that were frequently employed for analyzing the MDOF NSD systems. One is the equivalent linearization (EQL) method which was proposed by Booton in 1954 in his investigation on the control of electronic NSD systems [18]. The EQL method was widely investigated and employed thereafter in solving the real problems arising from science and engineering [19,20]. Another method applicable for analyzing MDOF NSD systems is the Monte Carlo simulation (MCS) method which is for the numerical solution of stochastic differential equations [21,22]. The EQL is based on the assumption that the system responses are Gaussian and hence the first and second moments of the system responses can be well estimated with EQL if the system nonlinearity is weak. The MCS can be employed for analyzing many NSD systems, but the computational effort needed by the MCS is huge, if affordable, when the system is large or nonlinearity is strong and the small probability of system responses is concerned. The numerical convergence, stability, round-off error, and requirement for large sample size are also challenges for the MCS method in analyzing the

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strongly nonlinear MDOF or large NSD systems. Recently a new method named state-space-split (SSS) method was proposed for analyzing the probabilistic solutions of the MDOF NSD systems excited by Gaussian white noise [23]. The SSS method was further extended to analyzing the MDOF NSD systems excited by Poissonian white noise [24]. With the SSS method, the FPK equation in high dimensions can be reduced to some approximate FPK equations in low dimensions. The approximate FPK equations in low dimensions can then be solved with the EPC method. Hence the whole solution procedure is named SSS–EPC method in the following discussion. With the SSS–EPC method, the system dimensions are not limited and the systems with strong nonlinearity can also be analyzed. It is known that the noises in the real world, such as the seismic ground motion, the wind velocity, the random sea wave motion, etc., are not white, but colored. Some of the real noises can be modeled as the response of the second-order differential oscillator excited by Gaussian white noise [25]. In this paper, the SSS–EPC method is further extended to analyzing the stationary probabilistic solutions of the MDOF NSD systems with polynomial type of nonlinearity and excited by the colored noise described by the response of second-order differential oscillator excited by Gaussian white noise to verify the effectiveness of the SSS–EPC method in this case.

2. Problem formulation

In the following discussion, the summation convention applies unless stated otherwise. The random state variable or vector is denoted with capital letter and the corresponding deterministic state variable or vector is denoted with the same letter but in lowercase.

Many problems in science and engineering can be described by the following MDOF NSD system.

$$\ddot{Y}_i + h_i(\mathbf{Y}, \dot{\mathbf{Y}}) = \beta_i F(t) \quad i, j = 1, 2, \ldots, n_\mathbf{y} \tag{1}$$

where $Y_i \in \mathbb{R}$ $(i = 1, 2, \ldots, n_\mathbf{y})$ are the components of the vector process $\mathbf{Y} \in \mathbb{R}^{n_\mathbf{y}}$; $h_i(\mathbf{Y}, \dot{\mathbf{Y}})$ are the polynomial type of nonlinear functions of $\mathbf{Y}$ and $\dot{\mathbf{Y}}$, $h_i : \mathbb{R}^{2n_\mathbf{y}} \to \mathbb{R}$; β_i $(i = 1, 2, \ldots, n_\mathbf{y})$, are constants; $F(t)$ is the excitation being filtered Gaussian white noise governed by the following stochastic differential equation.

$$\ddot{F}(t) + h(F, \dot{F}) = \alpha W(t) \tag{2}$$

where α is a constant; $h(F, \dot{F})$ is the polynomial type of linear or nonlinear function of F and $\dot{F}$, $h : \mathbb{R}^2 \to \mathbb{R}$; $W(t)$ is Gaussian white noise with zero mean and auto-correlation $E[W(t)W(t+\tau)] = S\delta(\tau)$ in which $\delta(\tau)$ is Dirac's delta function and S is a constant representing the power spectral density (PSD) of $W(t)$.

Setting $Y_i = X_{2i-1}$, $\dot{Y}_i = X_{2i}$, $f_{2i-1} = X_{2i}$, $f_{2i} = \beta_i F - h_i$, $g_{2i-1} = g_{2i} = 0$ $(i = 1, 2, \ldots, n_\mathbf{y})$, $F = X_{2n_\mathbf{y}+1}$, $\dot{F} = X_{2(n_\mathbf{y}+1)}$, $f_{2n_\mathbf{y}+1} = X_{2(n_\mathbf{y}+1)}$, $f_{2(n_\mathbf{y}+1)} = -h(X_{2n_\mathbf{y}+1}, X_{2n_\mathbf{y}+2})$, $g_{2n_\mathbf{y}+1} = 0$, $g_{2(n_\mathbf{y}+1)} = \alpha$, and $n_\mathbf{x} = 2(n_\mathbf{y}+1)$, then Eqs. (1) and (2) can be expressed by the following coupled Langevin equations or Ito differential equations.

$$\frac{d}{dt} X_i = f_i(\mathbf{X}) + g_i W(t) \quad i = 1, 2, \ldots, n_\mathbf{x} \tag{3}$$

where $\mathbf{X} \in \mathbb{R}^{n_\mathbf{x}}$; X_i $(i = 1, 2, \ldots, n_\mathbf{x})$, are the components of the state vector process $\mathbf{X}$; $f_i(\mathbf{X}) : \mathbb{R}^{n_\mathbf{x}} \to \mathbb{R}$.

The state vector process $\mathbf{X}$ is Markovian and the probability density function (PDF) $p(\mathbf{x}, t)$ of the Markovian vector process is governed by the FPK equation. When the white noise is Gaussian, the stationary PDF $p(\mathbf{x})$ of the Markovian vector is governed by the following reduced FPK equation [1].

$$\frac{\partial}{\partial x_i} [f_i(\mathbf{x}) p(\mathbf{x})] - \frac{\alpha^2 S}{2} \frac{\partial^2}{\partial x_{n_\mathbf{x}}^2} p(\mathbf{x}) = 0 \tag{4}$$

where $\mathbf{x}$ is the deterministic state vector, $\mathbf{x} \in \mathbb{R}^{n_\mathbf{x}}$.

It is assumed that the solution of Eq. (4) fulfills the following conditions:

$$\lim_{x_i \to \pm\infty} f_i(\mathbf{x}) p(\mathbf{x}) = 0 \quad i = 1, 2, \ldots, 2n_\mathbf{y} \tag{5}$$

which can be fulfilled by the responses of many real problems or dynamical systems.

3. State-space-split method

If the joint PDF of Y_i and $\dot{Y}_i$ is needed, the joint PDF of Y_i, $\dot{Y}_i$, F, and $\dot{F}$ must be obtained first in the case that $F(t)$ is governed by Eq. (2), which is explained in the end of this section. In order to obtain the joint PDF of Y_i, $\dot{Y}_i$, F, and $\dot{F}$, separate the state vector $\mathbf{X}$ into two parts as $\mathbf{X}_1 = \{Y_i, \dot{Y}_i, F, \dot{F}\} = \{X_{2i-1}, X_{2i}, X_{n_\mathbf{x}-1}, X_{n_\mathbf{x}}\} \in \mathbb{R}^4$ for $i = 1, 2, \ldots,$ or $n_\mathbf{y}$, and $\mathbf{X}_2 \in \mathbb{R}^{n_\mathbf{x}-4}$, i.e., $\mathbf{X} = \{\mathbf{X}_1, \mathbf{X}_2\} \in \mathbb{R}^{n_\mathbf{x}} = \mathbb{R}^{n_{\mathbf{x}_1}} \times \mathbb{R}^{n_{\mathbf{x}_2}}$ with $n_{\mathbf{x}_1} = 4$ and $n_{\mathbf{x}_2} = n_\mathbf{x} - 4$.

Denote the PDF of $\mathbf{X}_1$ as $p_1(\mathbf{x}_1)$. In order to obtain $p_1(\mathbf{x}_1)$, integrating both sides of Eq. (4) over $\mathbb{R}^{n_{\mathbf{x}_2}}$ gives

$$\int_{\mathbb{R}^{n_{\mathbf{x}_2}}} \frac{\partial}{\partial x_i} [f_i(\mathbf{x}) p(\mathbf{x})]\, d\mathbf{x}_2 - \frac{\alpha^2 S}{2} \int_{\mathbb{R}^{n_{\mathbf{x}_2}}} \frac{\partial^2 p(\mathbf{x})}{\partial x_{n_\mathbf{x}}^2} d\mathbf{x}_2 = 0. \tag{6}$$

Because of the conditions in Eq. (5), we have

$$\int_{\mathbb{R}^{n_{\mathbf{x}_2}}} \frac{\partial}{\partial x_i} [f_i(\mathbf{x}) p(\mathbf{x})]\, d\mathbf{x}_2 = 0 \quad x_i \in \mathbb{R}^{n_{\mathbf{x}_2}}. \tag{7}$$

Eq. (6) can then be written as

$$\int_{\mathbb{R}^{n_{\mathbf{x}_2}}} \frac{\partial}{\partial x_i} [f_i(\mathbf{x}) p(\mathbf{x})]\, d\mathbf{x}_2 - \frac{\alpha^2 S}{2} \int_{\mathbb{R}^{n_{\mathbf{x}_2}}} \frac{\partial^2 p(\mathbf{x})}{\partial x_{n_\mathbf{x}}^2} d\mathbf{x}_2 = 0 \quad x_i \in \mathbb{R}^{n_{\mathbf{x}_1}} \tag{8}$$

which can be equivalently written as

$$\int_{\mathbb{R}^{n_{\mathbf{x}_2}}} \frac{\partial}{\partial x_i} [f_i(\mathbf{x}) p(\mathbf{x})]\, d\mathbf{x}_2 - \frac{\alpha^2 S}{2} \frac{\partial^2}{\partial x_{n_\mathbf{x}}^2} \int_{\mathbb{R}^{n_{\mathbf{x}_2}}} p(\mathbf{x}) d\mathbf{x}_2 = 0 \quad x_i \in \mathbb{R}^{n_{\mathbf{x}_1}}. \tag{9}$$

Because

$$\int_{\mathbb{R}^{n_{\mathbf{x}_2}}} p(\mathbf{x}) d\mathbf{x}_2 = p_1(\mathbf{x}_1) \tag{10}$$

then Eq. (7) can be further written as

$$\int_{\mathbb{R}^{n_{\mathbf{x}_2}}} \frac{\partial}{\partial x_i} [f_i(\mathbf{x}) p(\mathbf{x})]\, d\mathbf{x}_2 - \frac{\alpha^2 S}{2} \frac{\partial^2 p_1(\mathbf{x}_1)}{\partial x_{n_\mathbf{x}}^2} = 0 \quad x_i \in \mathbb{R}^{n_{\mathbf{x}_1}}. \tag{11}$$

Clustering the terms purely in $\mathbf{x}_1$ in one part and the other terms in another part, then $f_i(\mathbf{x})$ is decomposed into two parts as

$$f_i(\mathbf{x}) = f_i^I(\mathbf{x}_1) + f_i^{II}(\mathbf{x}). \tag{12}$$

Substituting Eq. (12) into Eq. (11) and noting Eq. (10) gives

$$\frac{\partial}{\partial x_i} \left[f_i^I(\mathbf{x}_1) p_1(\mathbf{x}_1) + \int_{\mathbb{R}^{n_{\mathbf{x}_2}}} f_i^{II}(\mathbf{x}) p(\mathbf{x}) d\mathbf{x}_2 \right] - \frac{\alpha^2 S}{2} \frac{\partial^2 p_1(\mathbf{x}_1)}{\partial x_{n_\mathbf{x}}^2} = 0 \quad x_i \in \mathbb{R}^{n_{\mathbf{x}_1}}. \tag{13}$$

Denote $f_i^{II}(\mathbf{x}) = \sum_k f_i^{II}(\mathbf{x}_1, \mathbf{z}_k)$ in which $\mathbf{z}_k \in \mathbb{R}^{n_{\mathbf{z}_k}} \subset \mathbb{R}^{n_{\mathbf{x}_2}}$. $n_{\mathbf{z}_k}$ is the number of the state variables in $\mathbf{z}_k$. Then Eq. (13) is written as

$$\frac{\partial}{\partial x_i} \left[f_i^I(\mathbf{x}_1) p_1(\mathbf{x}_1) + \sum_k \int_{\mathbb{R}^{n_{\mathbf{z}_k}}} f_i^{II}(\mathbf{x}_1, \mathbf{z}_k) p_k(\mathbf{x}_1, \mathbf{z}_k) d\mathbf{z}_k \right] - \frac{\alpha^2 S}{2} \frac{\partial^2 p_1(\mathbf{x}_1)}{\partial x_{n_\mathbf{x}}^2} = 0 \quad x_i \in \mathbb{R}^{n_{\mathbf{x}_1}} \tag{14}$$