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An efficient numerical technique for the solution of nonlinear singular boundary value problems

Randhir Singh *, Jitendra Kumar [1]

Department of Mathematics, Indian Institute of Technology Kharagpur, Kharagpur -721302, India

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ABSTRACT

In this work, a new technique based on Green's function and the Adomian decomposition method (ADM) for solving nonlinear singular boundary value problems (SBVPs) is proposed. The technique relies on constructing Green's function before establishing the recursive scheme for the solution components. In contrast to the existing recursive schemes based on the ADM, the proposed technique avoids solving a sequence of transcendental equations for the undetermined coefficients. It approximates the solution in the form of a series with easily computable components. Additionally, the convergence analysis and the error estimate of the proposed method are supplemented. The reliability and efficiency of the proposed method are demonstrated by several numerical examples. The numerical results reveal that the proposed method is very efficient and accurate.

1. Introduction

Accurate and efficient numerical methods are often necessary for the solution of nonlinear singular boundary value problems (SBVPs) for ordinary differential equations. Such nonlinear SBVPs arise very frequently in many branches of applied mathematics and engineering such as chemical reactions, gas dynamics, electro-hydrodynamics, nuclear physics, atomic structures, atomic calculations, and in the study of positive radial solutions of nonlinear elliptic equations, physiological studies, in the study of steady-state oxygen diffusion in a spherical cell [1] and the distribution of heat sources in the human head [2].

The main objective of this work is to introduce an efficient numerical technique based on the recent work of Singh et al. [3], where the authors transformed original weakly singular problem with Dirichlet and Robin boundary conditions into an integral equation before establishing the recursive scheme for the approximate solution. Here, we consider the following class of strongly nonlinear SBVPs with Neumann and Robin boundary conditions [4–8]

$$\left.\begin{aligned} &u''(x) + \frac{\alpha}{x}u'(x) = f(x, u(x)), \quad 0 < x \le 1,\ \alpha \ge 1 \\ &u'(0) = 0, \qquad au(1) + bu'(1) = c, \end{aligned}\right\} \tag{1.1}$$

where $a > 0$, $b \ge 0$ and c are any finite real constants. It is also assumed that for every $(x, u) \in \{(0, 1] \times (0, \infty)\}$, the nonlinear function $f(x, u)$ as well as its partial derivative $\frac{\partial f}{\partial u}$ is continuous and the condition $\frac{\partial f}{\partial u} \ge 0$ be satisfied. Eq. (1.1) arises frequently in applied sciences and engineering. For $\alpha = 1$ and $f(x, u) = u^\gamma$, where γ is a physical constant, the above equation is used to study thermal explosions [9]. For $\alpha = 2$ and $f(x, u) = \frac{\theta u}{u+\kappa}$, $\theta > 0$, $\kappa > 0$, it has applications for finding the steady-state oxygen diffusion in a spherical cell with Michaelis–Menten uptake kinetics [1]. Duggan and Goodman [2] studied the equation (1.1) with $\alpha = 2$ and $f(x, u) = -\delta_1 e^{-\theta_1 u}$, $\theta_1 > 0$, $\delta_1 > 0$ to find the distribution of heat sources in the human head. Another application arises for $\alpha = 2$ in the theory of electro-hydrodynamics with $f(x, u) = \nu e^u$, where ν is a physical parameter [6].

The numerical study of the SBVPs arising in various physical models has attracted the attention of many authors [1–22] and many of the references therein. The major difficulty of solving Eq. (1.1) is because of its singular behavior at $x = 0$. A variety of methods have been applied to tackle such SBVPs, for example, the cubic spline and the finite difference methods [4,5,9,13,14,20]. Although, these numerical methods have many advantages, but a huge amount of computational work is needed which combines some root-finding techniques for obtaining accurate numerical solution especially for nonlinear SBVPs.

Recently, some newly developed numerical-approximate methods, such as the Adomian decomposition method (ADM), modified Adomian decomposition method (MADM) and Homotopy analysis method (HAM) [6,7,19,22], have been applied to obtain an

* Corresponding author. Tel.: +91 9733847644.
E-mail addresses: randhir.math@gmail.com (R. Singh), jkumar@maths.iitkgp.ernet.in (J. Kumar).
[1] Tel.: +91 3222 283664.

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approximate solution of the Eq. (1.1). It is well known that solving SBVPs (1.1) using ADM/MADM or HAM is always a computationally involved task as it requires computation of undetermined coefficients in a sequence of nonlinear algebraic or more difficult transcendental equations. Moreover, in some cases the undetermined coefficients may not be uniquely determined and this may be the major disadvantage of these methods for solving nonlinear SBVPs.

Most recently, the variational iteration method (VIM) and its modified versions have also been employed in the literature [8,21,23]. These methods give good approximations only for linear problems and nonlinear problems with nonlinearity of the form u^n, uu', u'^n . . . , etc. However, these methods fail to solve the equation when the nonlinear function is of the form e^u, $\ln(u)$, $\sin u$, $\sinh u$ . . . , etc., see Wazwaz and Rach [23] for more details. Nevertheless, applications of VIM for solving nonlinear problems are very restrictive.

1.1. Review of ADM

In this subsection, we briefly describe ADM for solving SBVPs (1.1). Recently, many researchers [6,7,17,18,22,24–29] have used the ADM for solving different scientific models. Adomian [27] asserted that the ADM provides an efficient and computationally suitable method for generating an approximate series solution for differential equations.

According to the ADM, the SBVPs (1.1) can be written in the operator form as

$$\mathcal{L}u(x) = f(x, u(x)), \tag{1.2}$$

where $\mathcal{L}$ is a linear second order differential operator defined by

$$\mathcal{L} = x^{-\alpha}\frac{d}{dx}\left[x^{\alpha}\frac{d}{dx}\right].$$

The inverse operator $\mathcal{L}^{-1}$ is given by

$$\mathcal{L}^{-1}[\cdot] = \int_0^x x^{-\alpha}\int_0^x x^{\alpha}[\cdot]dxdx. \tag{1.3}$$

Operating $\mathcal{L}^{-1}[\cdot]$ on both sides of (1.2) and using the condition $u'(0) = 0$, we obtain

$$u(x) = c_1 + \mathcal{L}^{-1}[f(x, u(x))], \tag{1.4}$$

where $u(0) = c_1 \neq 0$ is an unknown constant to be determined.

The solution $u(x)$ and the nonlinear function $f(x, u(x))$ are decomposed by an infinite series as

$$u(x) = \sum_{j=0}^{\infty} u_j(x) \quad \text{and} \quad f(x, u(x)) = \sum_{j=0}^{\infty} A_j, \tag{1.5}$$

where A_j, $j = 0, 1, 2, \ldots$ are Adomian's polynomials which can be generated for various classes of nonlinear functions with the formula given in [24,30] as:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[f\left(x, \sum_{k=0}^{\infty} u_k\lambda^k\right)\right]_{\lambda=0}, \quad j = 0, 1, 2, \ldots. \tag{1.6}$$

By substituting the series (1.5) into (1.4), we have

$$\sum_{j=0}^{\infty} u_j(x) = c_1 + \mathcal{L}^{-1}\left[\sum_{j=0}^{\infty} A_j\right]. \tag{1.7}$$

Upon comparing both sides of (1.7), the ADM is given by

$$\left.\begin{aligned} &u_0(x) = c_1, \\ &u_j(x) = \mathcal{L}^{-1}[A_{j-1}], \quad j \geq 1. \end{aligned}\right\} \tag{1.8}$$

The recursive scheme (1.8) will lead to the complete determination of the components $u_n(x, c_1)$ of the exact solution $u(x)$. The series solution of $u(x)$ follows immediately with the unknown constant c_1 which will be determined by using the boundary condition at $x = 1$. Hence, the n-term truncated approximate series solution is obtained as

$$\phi_n(x, c_1) = \sum_{j=0}^{n} u_j(x, c_1). \tag{1.9}$$

Several researchers [6,7,22,25,26,28,29] have used the ADM or MADM to solve nonlinear boundary value problems (BVPs) for ordinary differential equations. However, solving such nonlinear BVPs using ADM or MADM is always a computationally involved job since it requires the computation of unknown constants in a sequence of difficult transcendental equations which increases the computational work.

In order to avoid solving a sequence of difficult transcendental equations for two-point boundary value problems, we propose an improved decomposition method (IDM) to overcome the difficulties occurring in ADM or MADM for solving nonlinear SBVPs. The proposed IDM is based on Green's function and ADM. This technique relies on constructing Green's function before establishing the recursive scheme for the solution components. In contrast to the existing recursive schemes based on the ADM, the proposed IDM avoids solving a sequence of transcendental equations for the undetermined coefficients. An approximation of the solution is obtained in the form of a series with easily computable components. Additionally, the convergence analysis and the error estimate of the proposed method are supplemented. The reliability and efficiency of the proposed method are demonstrated by several numerical examples. The numerical results reveal that the proposed method is very efficient and accurate.

2. The improved decomposition method (IDM)

In this section, we propose a new approach based on Green's function and ADM for solving nonlinear SBVPs (1.1). To this end, we consider the homogeneous version of (1.1) as follows:

$$\begin{cases} (x^{\alpha}u'(x))' = 0, & 0 < x \leq 1, \\ u'(0) = 0, & au(1) + bu'(1) = c. \end{cases} \tag{2.1}$$

Its unique exact solution is given by

$$\hat{u}(x) = \frac{c}{a}. \tag{2.2}$$

We again rewrite SBVPs (1.1) with homogeneous boundary conditions as

$$\begin{cases} (x^{\alpha}u'(x))' = x^{\alpha}f(x, u(x)), & 0 < x \leq 1, \\ u'(0) = 0, & au(1) + bu'(1) = 0. \end{cases} \tag{2.3}$$

Green's function of the problem (2.3) can easily be obtained as follows:

$$G(x, \xi) = \begin{cases} \ln\xi - \dfrac{b}{a}, & 0 < x \leq \xi \leq 1, \ \text{for } \alpha = 1 \\ \ln x - \dfrac{b}{a}, & 0 < \xi \leq x \leq 1, \end{cases} \tag{2.4}$$

and

$$G(x, \xi) = \begin{cases} \dfrac{\xi^{1-\alpha} - 1}{1-\alpha} - \dfrac{b}{a}, & 0 < x \leq \xi \leq 1, \ \text{for } \alpha > 1 \\ \dfrac{x^{1-\alpha} - 1}{1-\alpha} - \dfrac{b}{a}, & 0 < \xi \leq x \leq 1. \end{cases} \tag{2.5}$$

The derivation of Green's function is provided in the Appendix.