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A variational framework for high-order mesh generation

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Abstract

The generation of sufficiently high quality unstructured high-order meshes remains a significant obstacle in the adoption of high-order methods. However, there is little consensus on which approach is the most robust, fastest and produces the 'best' meshes. We aim to provide a route to investigate this question, by examining popular high-order mesh generation methods in the context of an efficient variational framework for the generation of curvilinear meshes. By considering previous works in a variational form, we are able to compare their characteristics and study their robustness. Alongside a description of the theory and practical implementation details, including an efficient multi-threading parallelisation strategy, we demonstrate the effectiveness of the framework, showing how it can be used for both mesh quality optimisation and untangling of invalid meshes.

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1. Introduction

The high accuracy and low dispersion and diffusion errors of high-order methods makes them ideal candidates for the unsteady simulations in areas such as aeroacoustics, turbulent flow and combustion. However, it is widely accepted that the lack of robust high-order mesh generators for complex geometries is still a major bottleneck in the wider adoption of these methods within industrial practice [1,2]. The generation of a curvilinear unstructured mesh is achieved in general through the transformation of a coarse straight-sided linear mesh, which can be obtained by any of the well-established unstructured mesh generators, onto a boundary conforming high-order mesh. The challenge in this approach is accommodating boundary curvature into the mesh interior so that the resulting curvilinear elements are valid; that is, they do not self-intersect. A secondary problem is generating meshes that are of sufficient quality to allow simulations that retain the accuracy and convergence properties of the underlying high-order discretisation.

As high-order methods have gained interest within the community, there is an increasing number of approaches used to achieve this linear-to-curvilinear transformation. They can be broadly classified into two groups: optimisation of a functional in which distortion of elements or a mesh energy is minimised, and solid body formulations where the mesh is modelled as an elastic body and an elliptic PDE is then solved to obtain a displacement within the domain. In the first category, Dey et al. [3] investigated the use of the scaled Jacobian as a distortion metric to drive local

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curvature-based refinement and mesh optimisation. Sherwin and Peiro [4] use a spring analogy to position points in curves and surfaces to minimize a deformation energy. Toulorge et al [5] investigate the use of a logarithmic barrier technique in combination with an unconstrained optimisation of the Jacobian and theoretical bounds obtained through Bézier functions. A number of works by Roca and collaborators have investigated the use of shape distortion optimisation to produce unstructured curvilinear meshes, e.g. [6–8]. In the category of solid body deformation, there have been a similar number of investigations: a linear elasticity approach is used by Xie et al. [9], a non-linear elasticity by Persson and Peraire [10], and a Winslow formulation by Fortunato and Persson [11]. Additionally, some of the authors have also investigated a thermo-elastic model [12] in order to control the quality generated by elastic models.

The purpose of this work is to demonstrate that many of these approaches can be reformulated in a generalised framework, which is based on a variational approach to curvilinear mesh generation. In a variational setting, a functional defining a measure of energy over the mesh, and which takes as its arguments the mesh displacement and its derivatives, is minimised using a nonlinear optimisation strategy. We will show how this framework can be used for the purposes of both untangling invalid curvilinear meshes and for mesh optimisation of existing valid meshes.

The motivation of this study comes in the development of similar approaches in the linear mesh generation community, for example in references [13,14]. These have been under development as early as the 1970s, where Felippa investigated the applicability of direct energy searches to the problem of mesh generation [15]. However, these methods are yet to be fully investigated in the context of high-order mesh generation and optimisation, although they have been briefly mentioned in previous work [16]. In this variational framework, we can not only consider and compare many of the existing approaches to high-order mesh generation, but capitalise on a number of mathematical and technical advantages that this framework offers. Firstly, we may examine a number of different functionals and compare their behaviour in terms of speed of convergence and the resulting mesh quality obtained. From the standpoint of analysis, the use of an energy functional guarantees the existence of a minimum under certain conditions of the behaviour of the functional such as convexity or polyconvexity, as noted by Evans [17]. Although we do not study this property in detail here (see, for example, Huang and Russell [14] and Garanzha [18]), this approach adds robustness to the method through these theoretical guarantees. From a technical perspective, the implementation of a nonlinear optimisation strategy for these functionals is arguably easier than, for example, a finite element discretisation of linear or nonlinear PDEs. We will show how this variational formulation permits highly efficient and effective multi-threading parallel execution, thus allowing for the optimisation of large high-order meshes in minutes on a modern desktop workstation.

The paper is structured as follows. In section 2, we give a brief formulation of the problem in terms of the underlying solid mechanics formulation and outline the four energy functionals that we will investigate in this work, which overlap with a large number of studies based around high-order mesh generation. In section 3 we describe details of the practical implementation needed in this variational setting such as discretisation and non-linear optimisation, with a brief discussion on untangling in section 4. Section 5 then examines the application of this method to a number of two- and three-dimensional problems, examining the meshes obtained by each method, the number of iterations and computational time needed for convergence. We finalise the paper in section 6 with a brief overview and outlook to future work and improvements.

2. Background and formulation

The premise of this work is to examine the generation of a boundary-conforming high-order mesh that is obtained through a deformation. The starting point is a straight-sided high-order mesh $\Omega = \bigcup_{e=1}^{N_{\mathrm{el}}} \Omega^e$ composed of N_{el} conformal elements, which is equipped with geometry-conforming displacements at the boundary of the domain. We view Ω as a solid body, so that the displacements at the boundary induce a deformation in the interior of the domain. The end result should be a valid high-order mesh Ω_*, where the interior elements have been deformed to accommodate the displacement at the boundary and improve the quality of the resulting elements.

From a solid mechanics perspective, we may construct a model that, for example, represents linear or nonlinear elasticity. This often appears in the form of an elliptic partial differential equation which may be solved to calculate the displacement $\boldsymbol{u}$ between the original and deformed states. As mentioned in the introduction, these approaches have both previously been examined in the context of high-order mesh generation. However the purpose of this work