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Synchronization-based Estimation of the Maximal Lyapunov Exponent of Nonsmooth Systems

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Abstract

The maximal Lyapunov exponent of a nonsmooth system is the lower bound for the proportional feedback gain necessary to achieve full state synchronization. In this paper, we prove this statement for the general class of nonsmooth systems in the framework of measure differential inclusions. The results are used to estimate the maximal Lyapunov exponent using chaos synchronization, which is illustrated on an impact oscillator.

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1. Introduction

The spectrum of Lyapunov exponents is an important characteristic of limit sets. It measures the exponential convergence or divergence of nearby trajectories, thereby capturing the sensitivity of solutions with respect to initial conditions[1]. An infinitesimal sphere of perturbed initial conditions will deform into an ellipsoid under the flow of a smooth dynamical system[2]. The Lyapunov exponents capture the average exponential growth or decay rate of the principal axes of the ellipsoid and the maximal Lyapunov exponent captures the long-term behavior of the dominating direction. A positive maximal Lyapunov exponent implies instability of the limit set (i.e., equilibrium, limit cycle, periodic or quasi-periodic solution) or it can be an indication for a chaotic attractor[3,4].

The existence of the Lyapunov exponents is a subtle question for non-conservative systems[5,6]. The mathematic foundation for the existence is given by the multiplicative ergodic theorem of Oseledec[7,8]. It states that, if there exists an invariant measure of the flow, then the Lyapunov exponents exist for almost every point w.r.t. that measure.

Algorithms to find the spectrum of Lyapunov exponents of smooth systems are well established[9–11]. The spectrum can be computed numerically by linearizing the differential equations along the nominal solution. Time integration of the linearized equations yields the fundamental solution matrix from which the spectrum can be obtained. Lyapunov exponents can also be obtained from experimental time series of systems with unknown dynamics[12–15].

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Dynamical systems with a discontinuous right-hand side exhibit discontinuities in the evolution of the fundamental solution matrix. The jumps can be described using a saltation matrix [16], which has been used for example for the numerical computation of the Lyapunov exponents with jump conditions including the motion on sliding surfaces [17] or for Filippov-type systems with an emphasis on mechanical systems with Coulomb friction [18]. A model based algorithm for the calculation of the spectrum of Lyapunov exponents has been developed for dynamical systems with discontinuous motion [19].

Two diffusively coupled identical smooth systems achieve synchronization despite the complicated dynamics of the individual systems if the coupling parameter is large enough [20]. The minimal value of the coupling parameter for which the synchronization set is (attractively) stable is determined by the maximal Lyapunov exponent of the individual systems. This relation arises from the competitive behavior of the separation due to the trajectory instability (dominated by the maximal Lyapunov exponent) and the convergence due to the coupling. Using this relation, the maximal Lyapunov exponent can be estimated by the critical coupling necessary for synchronization [21]. The method of estimating the maximal Lyapunov exponent using chaos synchronization has been considered for nonsmooth systems [22] and for discrete maps [23] assuming that the increase of the initial perturbation is uniform in time, which is only the case for linear time-invariant systems).

In this paper, we consider the class of nonsmooth systems with solutions of special locally bounded variation, which can be written in the framework of measure differential inclusions [24]. We prove for this general class of nonsmooth systems that the critical coupling is indeed given by the maximal Lyapunov exponent as long as it exists. The paper is organized as follows. We first restrict ourselves to smooth systems in Section 2 before we state the main result for nonsmooth systems in Section 3. The results are illustrated in Section 4 on an impact oscillator and conclusions are given in Section 5.

2. Smooth systems

The dynamics of a smooth system is given by

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{f}(\boldsymbol{x}, t), \tag{1}$$

where the vector field $\boldsymbol{f} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is continuously differentiable in its first argument and continuous in its second argument. We denote the solution of (1) for the initial conditions $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ as $\boldsymbol{x}(t) = \boldsymbol{\varphi}(t, \boldsymbol{x}_0, t_0)$, where the dependence on initial conditions is written explicitly. We introduce the perturbed solution $(\boldsymbol{x} + \Delta\boldsymbol{x})(t) = \boldsymbol{\varphi}(t, \boldsymbol{x}_0 + \kappa\boldsymbol{e}, t_0)$ obtained using the perturbed initial conditions $\boldsymbol{x}_0 + \kappa\boldsymbol{e}$ with $\|\boldsymbol{e}\| = 1$ and $\kappa > 0$ small. The dynamics of the perturbation $\Delta\boldsymbol{x}(t)$ is obtained as

$$\frac{\mathrm{d}\,(\Delta\boldsymbol{x})}{\mathrm{d}t} = \boldsymbol{f}(\boldsymbol{x} + \Delta\boldsymbol{x}, t) - \boldsymbol{f}(\boldsymbol{x}, t) = \boldsymbol{A}(t)\Delta\boldsymbol{x} + \boldsymbol{o}\left(\|\Delta\boldsymbol{x}\|\right), \tag{2}$$

where $\boldsymbol{A}(t) := \left.\frac{\partial \boldsymbol{f}(\boldsymbol{x},t)}{\partial \boldsymbol{x}}\right|_{\boldsymbol{\varphi}(t,\boldsymbol{x}_0,t_0)}$ is the linearization of the vector field $\boldsymbol{f}$ along the unperturbed solution and $\boldsymbol{o}$ denotes the (small) Landau-order symbol. The perturbation $\Delta\boldsymbol{x}$ tends to zero for $\kappa \to 0$. Therefore, we introduce the normalized perturbation $\boldsymbol{\xi}(t, \boldsymbol{e}, t_0) := \lim_{\kappa\to 0} \boldsymbol{\xi}_\kappa(t, \boldsymbol{e}, t_0)$, where $\boldsymbol{\xi}_\kappa(t, \boldsymbol{e}, t_0) := \frac{\Delta\boldsymbol{x}}{\|\Delta\boldsymbol{x}_0\|} = \frac{\boldsymbol{\varphi}(t,\boldsymbol{x}_0+\kappa\boldsymbol{e},t_0)-\boldsymbol{\varphi}(t,\boldsymbol{x}_0,t_0)}{\kappa}$. The limit exists because $\boldsymbol{O}\left(\|\Delta\boldsymbol{x}\|\right) = \boldsymbol{O}\left(\|\Delta\boldsymbol{x}(t_0)\|\right) = \boldsymbol{O}\left(\kappa\right)$, where $\boldsymbol{O}$ denotes the (big) Landau-order symbol. Taking the limit $\kappa \to 0$ of (2), divided by κ, yields

$$\lim_{\kappa\to 0} \frac{\mathrm{d}\boldsymbol{\xi}_\kappa}{\mathrm{d}t} = \boldsymbol{A}(t)\boldsymbol{\xi}. \tag{3}$$

The vector field $\boldsymbol{f}$ is continuously differentiable in its first argument, which implies local uniform convergence of $\lim_{\kappa\to 0} \frac{\mathrm{d}\boldsymbol{\xi}_\kappa}{\mathrm{d}t}$. Using Theorem 7.17 of Rudin [25] together with (3) yields

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} = \boldsymbol{A}(t)\boldsymbol{\xi}. \tag{4}$$

Let $\boldsymbol{\Phi}(t, t_0)$ be the fundamental solution matrix, which is the solution to the matrix differential equation $\frac{\mathrm{d}\boldsymbol{\Phi}}{\mathrm{d}t} = \boldsymbol{A}(t)\boldsymbol{\Phi}$ for the initial conditions $\boldsymbol{\Phi}(t_0, t_0) = \boldsymbol{I}$, where $\boldsymbol{I}$ is the identity matrix. Then, the solution of the normalized perturbation $\boldsymbol{\xi}$