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Stability analysis of the wave equation with delayed boundary conditions

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Abstract

Stability of the 1D wave equation with delayed boundary conditions is considered. By means of the traveling wave solution, the original system is transformed into a delay differential equation of neutral type which involves two delays. The complete and exact stability chart is presented for the delayed boundary problem in the parameter plane of the feedback gain between the boundaries and the ratio of the time delays.

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Keywords: wave equation; delayed feedback; delayed boundary condition; stability chart

1. Introduction

The 1D wave equation is a second-order linear partial differential equation

$$\ddot{\phi}(x,t) - c^2\phi''(x,t) = 0, \tag{1}$$

where $\phi(x,t)$ describes the displacement or rotational angle at spatial position x at time t, and c denotes the speed of wave. This equation serves as an important mathematical model for the study of continuum dynamical systems. For example, longitudinal vibration of a beam [1], torsional vibration of a shaft and transverse vibration of a taut string [2] can be modeled by the 1D wave equation (1).

In practical applications, feedback loops are often used to reduce noise and/or vibration in continuum dynamical systems. However, delay arises in any feedback loop. The presence of this unavoidable time delay in the controlled continuum system may lead to governing equations that consist of partial differential equations with delayed boundary conditions. For example, Ma and Butcher [3] investigated the delayed follower force applied on a continuum beam, while Haraguchi and Hu [4]studied the vibration suppression of noise of an acoustic system in the presence of delayed control. In both cases, they got the partial differential equations with delayed boundary conditions, which were too

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complex for analytical approaches. Therefore, they used low DoF approximation, a common technique when dealing with the continuum dynamical systems. However, as the number of degrees of freedom (DoF) increases, difficulties may arise whether the finite DoF approximation tends to the continuum model[5]. Kidd and Stepan[6] carried out a study on a basic microphone-amplifier-loudspeaker system which was modeled as a linear elastic beam with time delay in the control loop. It resulted in a wave equation with delayed boundary conditions. In this case, finite, even large DoF approximations were applied and it did not show convergence in the stability charts for certain small stable domains that were found numerically in the delayed continuum model.

The unavoidable occurrence of even a small time delay often greatly complicates the characteristics of the stability of the original dynamical systems[7]. Datko found that feedback stabilized hyperbolic systems were not robust with respect to small delays[8]. Hale and Lunel's mathematical result[9] showed that the linear difference equations with two discrete delays are almost always unstable when the coefficients are large enough. These difference equations are the essential parts of the neutral delay differential equations (NDDE), and consequently, undamped continuum systems with delayed boundaries can be unstable for any irrational ratios of the arising delays. Still, a few zero-probability stability domains were found in some numerical examples, while these tiny stable domains could not be obtained through finite DoF approximations[6]. Therefore, in spite of the complexity of delayed boundary problems of continuum systems, it is of great importance to attract efforts for analytical study of the original infinite dimensional systems.

The aim of this study is to determine the full stability chart in closed analytical form of the 1D wave equation when the following boundary conditions are considered.

$$\phi(l, t) = 0\,, \tag{2}$$

$$\phi'(0, t) - K\phi'(l, t - \tau) = 0\,. \tag{3}$$

where l is the length of the medium, K is the feedback gain and τ is the unavoidable time delay that arises in the feedback loop.

2. Traveling Wave Solution

The traveling wave solution of Eq.(1) takes the following form:

$$\phi(x, t) = f(t - x/c) + g(t + x/c)\,, \tag{4}$$

where f and g are unknown scalar functions. The time needed for a wave of speed c traveling along the medium is denoted by $T = l/c$. By substituting the traveling wave solution (4) into (2) and (3), the boundary conditions are expressed as follows:

$$f(t - T) + g(t + T) = 0\,, \tag{5}$$

$$(-f'(t) + g'(t)) - K(-f'(t - \tau - T) + g'(t - \tau + T)) = 0\,. \tag{6}$$

Eq. (5) results in a relation between g and f:

$$g(t) = -f(t - 2T)\,, \tag{7}$$

with which Eq. (6) can be transformed into an NDDE with respect to f in the form:

$$f'(t) + f'(t - 2T) - 2Kf'(t - T - \tau) = 0\,. \tag{8}$$

After substituting of the exponential trial function $f(t) = Be^{\lambda t}$, the corresponding characteristic function $D(\lambda)$ is obtained as follows:

$$D(\lambda) = \lambda(1 + e^{-2T\lambda} - 2Ke^{-(T+\tau)\lambda})\,. \tag{9}$$