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Technical note

# A mathematical method for precisely calculating the radiographic angles of the cup after total hip arthroplasty

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A B S T R A C T

We established a mathematical method to precisely calculate the radiographic anteversion (RA) and radiographic inclination (RI) angles of the acetabular cup based on anterior–posterior (AP) pelvic radiographs after total hip arthroplasty. Using Mathematica software, a mathematical model for an oblique cone was established to simulate how AP pelvic radiographs are obtained and to address the relationship between the two-dimensional and three-dimensional geometry of the opening circle of the cup. In this model, the vertex was the X-ray beam source, and the generatrix was the ellipse in radiographs projected from the opening circle of the acetabular cup. Using this model, we established a series of mathematical formulas to reveal the differences between the true RA and RI cup angles and the measurements results achieved using traditional methods and AP pelvic radiographs and to precisely calculate the RA and RI cup angles based on post-operative AP pelvic radiographs. Statistical analysis indicated that traditional methods should be used with caution if traditional measurements methods are used to calculate the RA and RI cup angles with AP pelvic radiograph. The entire calculation process could be performed by an orthopedic surgeon with mathematical knowledge of basic matrix and vector equations.



## 1. Introduction

An accurate assessment of the orientation of the acetabular cup is important for evaluating the post-operative outcome of total hip arthroplasty (THA) and improving surgical methods. An incorrect orientation of an acetabular component is closely correlated with a limited post-operative range of hip motion, hip dislocation, accelerated component wear and loosening as well as other systematic side-effects, such as high serum metal ion levels [1–3].

A variety of measurement methods have been used to determine the radiographic angles of the acetabular cup, which are mainly based on anterior–posterior (AP) hip radiographs [1–9]. Although previous studies on this topic have introduced many types of mathematical and trigonometric computations and geometrical analyses, a required precondition is that the X-ray's beam source is directed at the acetabular cup's center, which is how standard AP hip radiographs are taken. If the X-ray beam is directed at the pubic symphysis or another arbitrary area rather than the hip center when routine AP pelvic radiographs or unstandardized AP hip radiographs are taken, the acetabular cup will be obliquely projected, and the ellipse projecting from the opening circle of the cup on the X-ray film will be different from a standard AP hip radiograph. In this situation, most previous methods are not suitable but such methods are still used in some studies [9–13].

Although some researchers have reported differences between measurements of cup orientation based on pelvic and hip AP radiographs [2,5,6,14,15] and others have proposed corrective methods to reduce or eliminate the measurement errors [16–18], the methods and results vary between studies. Thus, a new calculation technique is needed to precisely calculate the orientation of the acetabular cup based on plain post-operative radiographs.

In this study a mathematical model for an oblique cone was established to simulate how AP pelvic radiographs are obtained and to address the relationship between the two-dimensional (2D) and three-dimensional (3D) geometry of the opening circle of the cup.

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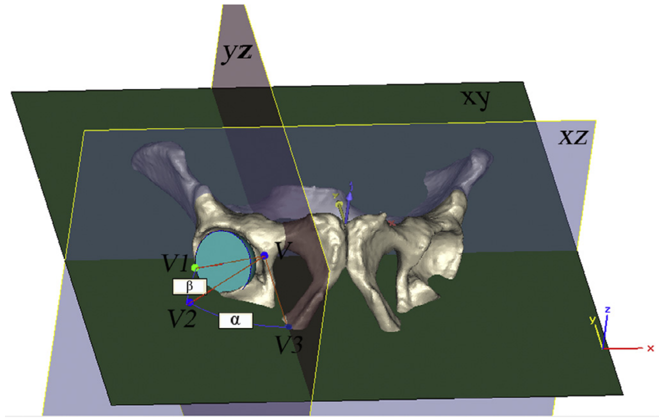


**Fig. 1.** The radiographic anteversion (RA) and radiographic inclination (RI) angles according to Murray et al.'s definition. The RA angle is defined as the intersection angle between the axis of the acetabular cup and its projection onto the body's coronal plane, and the RI angle is the intersection angle between the longitudinal axis of the body and the projection of the axis of the acetabular cup onto the coronal plane. Plane *xy* represents the body's coronal plane. Line *VV1* represents the axis of the acetabular cup, the direction of which is equal to the normal vector of the axis of the acetabular cup, line *VV2* represents the projection of the axis of the acetabular cup onto the *xy* plane and line *VV3* represents the longitudinal axis of the body. The RI and RA angles are denoted as $\alpha$ and $\beta$, respectively.

In this model, the vertex was the X-ray beam source, and the generatrix was the ellipse projected from the opening circle of the cup on radiographs. Using the oblique cone model, we could determine the differences between the true radiographic angles of the cup in 3D space and the measured values using previous methods based on pelvic or unstandardized acetabular AP radiographs and to precisely calculate the true radiographic anteversion (RA) and radiographic inclination (RI) angles based on post-operative AP pelvic radiographs.

## 2. Methods

The study was approved by the Ethics Committee of the Chinese PLA General Hospital. The patient's informed consent was obtained in advance to the collection and use of CT data for 3D reconstruction.

Fig. 1 shows the definition of the RA and RI angles of the acetabular cup, according to Murray [19]. The established computational processes for calculating RA and RI angles were performed using Mathematica software 8 (Wolfram Research Inc., Champaign, IL, USA) and consisted of two main parts: (1) determining the projected ellipse based on different combinations of true RA and RI cup angles; and (2) calculating the true RA and RI cup angles based on the projected ellipse from the AP pelvic radiographs.

### 2.1. Projection from circle to ellipse

The calculating procedures for revealing differences between the RA and RI angles measured in an AP radiograph using traditional methods and the corresponding true angles are described here in detail. Before the calculation procedures are performed, a coordinate system is first established to simulate how AP pelvic radiographs are collected. This process is also suitable for unstandardized AP hip radiographs, in which the X-ray beam is not directed at the center of the hip. In this system, the radiograph is defined as the *xy* plane, and the center of the radiograph is set as the original point (Fig. 2). We know that when the AP radiograph is collected, the X-ray beam is directed at the center of the radiograph. Thus, the X-ray beam source is on the *z*-axis (Fig. 2). The distance between the X-ray beam source and the radiograph plane can be retrieved from the radiographic staff or from the properties of the electronic radiograph in Digital Imaging and Communication in Medicine (DICOM) format, allowing determination of the X-ray beam source coordinates.

If the normal vector of the axis of the acetabular cup is $[k, i, j]$ and the RI and RA angles are $\alpha$ and $\beta$, respectively, according the definition by Murray, the cosines of the RI and RA angles can be calculated as follows.

$$\cos(\alpha) = \frac{i}{\sqrt{k^2 + i^2}} \tag{1}$$

$$\cos(\beta) = \frac{k^2 + i^2}{\sqrt{k^2 + i^2}} \tag{2}$$

If the RA and RI angles are known, the normal vector of the axis of the acetabular cup (equal to the normal vector of Line *VV1* in Fig. 1) can be calculated and vice versa.

To determine the ellipse projected by the X-ray from the opening circle of the acetabular cup in the X-ray imaging plane (ellipse *Eu3* in Fig. 2), we first need to establish an equation for the opening circle of the cup. The equation for a space circle usually consists of two equations for a sphere and a plane in which the space circle is located.

The equation for a plane in which the opening circle of the cup is located, can be established based on the normal vector of the axis of the cup and the coordinate of the center of the opening circle of the cup, as the normal vector of this plane is the same as that of the axis of the cup. The equation for a sphere can be established based on the center and the radius of the opening circle of the cup.

After the equation for the opening circle of the cup is established using simultaneous equations for the plane and the sphere, the space line connecting the X-ray beam source and the points located on the opening circle of the cup can be determined accordingly.

As the general equation for an ellipse has six coefficients, the calculation for the ellipse projected from the opening circle of the cup can be determined after six points on the opening circle of the cup (points A'–F' in Fig. 2b) are specified using the equations for the opening circle of the cup. Then, all six of the point-to-point coordinates on the ellipse projected from the opening circle of the cup in the X-ray film can be determined using the equations for the space line connecting the X-ray beam source and the points on the opening circle of the cup. Based on these six points on the projected ellipse (points A–F in Fig. 2b), the equation for the projected ellipse can be established based on the following general equation for the ellipse (as shown in Fig. 3).

$$ax^2 + by^2 + c + 2fy + 2gx + 2hxy = 0 \tag{3}$$

After the rotation and translation manipulations of the coordinate system, the standard equation for the ellipse (4) can be transformed and developed from the general Eq. (3), and the short semi-axis, $a_1$, and long semi-axis, $b_1$, can be calculated as follows.

$$\frac{(x - x_2)^2}{{a_1}^2} + \frac{(y - y_2)^2}{{b_1}^2} = 1 \tag{4}$$

Based on the traditional method for calculating the RA angle in X-ray film used in previous studies [1–9], the RA angle of the cup is the arcsine of $a_1/b_1$. The RI angle measured in the X-ray film is equal to the rotation angle of the coordinate system that was used to deduce the standard equation for the ellipse.

The detailed deduction process for this section is provided in Supplementary Material Part A.

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