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Methodological and Ideological Options

Applications of aggregation theory to sustainability assessment☆

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ABSTRACT

In order to aid operations that promote sustainability goals, researchers and stakeholders use sustainability assessments. Although assessments take various forms, many utilize diverse sets of indicators numbering anywhere from two to over 2000. Indices, composite indicators, or aggregate values are used to simplify high dimensional and complex data sets and to clarify assessment results. Although the choice of aggregation function is a key component in the development of the assessment, there are few literature examples to guide appropriate aggregation function selection. This paper applies the mathematical study of aggregation functions to sustainability assessment in order to aid in providing criteria for aggregation function selection. Relevant mathematical properties of aggregation functions are presented and interpreted. Cases of these properties and their relation to previous sustainability assessment research are provided. Examples show that mathematical aggregation properties can be used to address the topics of compensatory behavior and weak versus strong sustainability, aggregation of data under varying units of measurements, multiple site multiple indicator aggregation, and the determination of error bounds in aggregate output for normalized and non-normalized indicator measures.

1. Introduction

A challenge for assessing sustainability is that it is not a single entity that can be readily measured. Instead sustainability is a combination of several aspects of the physical and biotic environment, social welfare, and economic wellbeing. Furthermore, it is an aspiration rather than a state. Its meaning is largely determined by contextual circumstances (Efroymson et al., 2013). Yet it is important to be able to measure, quantify, and discuss progress toward that goal.

Current sustainability assessment approaches often represent sustainability using multiple indicators, multiple variables, or multiple data points. At a minimum, the consensus is that sustainability needs to incorporate environmental, social, and economic conditions, which are referred to as the three pillars of sustainability (Mori and Christodoulou, 2012; Hacking and Guthrie, 2008; Mayer, 2008; Brundtland and World Commission on Environment and Development, 1987). In practice, sustainability indices can incorporate data from over 2600 indicator variables (The Living Planet Index, (McRae et al., 2012)). To add further complexity, each input variable often has an associated data set containing multiple observations. These large amounts of data about diverse components of sustainability are difficult to manage and nearly impossible to visualize without some sort of compression or reduction of dimensionality.

Aggregation functions are one method employed to accomplish this task of clarifying and simplifying data. Aggregation theory is the area of mathematics that explores the form and properties of such aggregation functions. In ecological economics the topic of aggregation comes up in regard to spatial aggregation (Su and Ang, 2010), valuation of ecosystem benefits (Tait et al., 2012; Lele and Srinivasan, 2013), calculation of conservation benefits (Winands et al., 2013), and combining information across sectors (Lenzen, 2007; Marin et al., 2012).

This study introduces basic properties, definitions, and theory related to the process of aggregation in order to aid in providing a rigorous mathematical baseline for further development of sustainability assessment techniques and methodologies. This paper deals with the conditions that must be met in order for information to be combined in an accurate, consistent, and overall robust manner. Five examples highlight some of the many relationships that can be derived between mathematical aggregation theory and sustainability assessment. These examples include mathematical interpretation of weak and strong sustainability, a proof that provides a simple bound for aggregate outputs under varying levels of relative error using the arithmetic mean, and two examples

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of how grouping and aggregation can lead to inconsistent results depending on how aggregation takes place. The final section discusses multiple invariance properties with respect to the scale of measurability of the indicators to be aggregated and includes an example of how a simple change in measurement units can create inconsistent aggregate outputs. The 2004 paper by Ebert and Welsch, which provides a guideline for choosing aggregation functions, is interpreted and placed into the larger mathematical aggregation theoretic context.

2. Basic Properties of Aggregation Functions

The process of aggregation is ubiquitous in the sciences. However, the word aggregation can take on different meanings within different disciplines. The book *Aggregation Functions* (Grabisch et al., 2009) presents a comprehensive mathematical treatment of aggregation functions and their properties and is a unique resource within the mathematics literature. The definitions provided in Grabisch et al. (2009) are adopted here. Beginning with the formal definition for an aggregation function, the following section establishes the basic terms and properties used. For each set of properties presented, a mathematical definition is provided along with interpretations related to sustainability assessment to provide context.

2.1. Definition of an Aggregation Function

In general, an aggregate value is a single representative value for an arbitrarily long set of related values. An aggregation function is the mathematical operation that maps the input values to the representative output value or 'aggregate'. Formally, for some nonempty real interval $\mathbb{I} \subseteq \mathbb{R}$ containing the values to be aggregated, an *aggregation function* in $\mathbb{I}^n$ is a function:

$$A^{(n)} : \mathbb{I}^n \rightarrow \mathbb{I}$$

that

(i) is nondecreasing (in each variable)
(ii) fulfills the following boundary conditions:

$$\inf_{\mathbf{x} \in \mathbb{I}^n} A^{(n)}(x) = \inf \mathbb{I} \text{ and } \sup_{\mathbf{x} \in \mathbb{I}^n} A^{(n)}(x) = \sup \mathbb{I} \quad (1)$$

where n represents the number of variables in the argument of the function, that is, the number of values to be aggregated or the dimension of the input vector, x. In general, an aggregation function $A^{(n)}(x)$ is written as $A(x)$ with the number of variables in its argument suppressed. Also note that the domain associated with a given aggregation function often changes with assessment context.

As an interpretation, condition (i) states that if any input value increases, the aggregate output value cannot decrease. Condition (ii) dictates what must happen at the boundary values. For example, if a set of indicators are normalized to values between 0 and 1, then the nonempty interval is given by $\mathbb{I} = [0, 1]$, and an aggregation function $A^{(n)}(x)$ must satisfy $A^{(n)}((0,..., 0)) = 0$ and $A^{(n)}((1,..., 1)) = 1$.

Table 1 gives some common aggregation functions and their definitions. The aggregation functions most frequently used in practice for sustainability assessment are the arithmetic and weighted arithmetic means (Singh et al., 2009; Böhringer and Jochem, 2007). Although the mathematical properties used to describe function behavior are numerous, certain properties of functions have particular importance to aggregation and are included here. The properties presented may help determine appropriate choices of aggregation functions given the sustainability indicator variables selected and the intended use within the assessment. Some of the mathematical definitions and properties presented, such as continuity, are familiar to mathematicians, while others, such as internality, conjunctivity, and disjunctivity as well as some of the grouping-based properties, are less familiar. However, within the context of sustainability assessment and aggregation theory, even familiar properties of functions can take on new meanings. The function property definitions in this paper follow the format of Grabisch et al. (2009), and interpretations relevant to sustainability assessment are provided when possible. Examples relating selected properties to sustainability assessment follow each set of properties provided.

2.2. Continuity Properties

Continuity relates closeness in the input variable(s) to closeness in the output variable(s) where closeness is defined using a specified norm. As such, continuity is important for understanding how the aggregation function performs with variable data or noise. Stronger and weaker forms of continuity exist. A strong form, Lipschitz continuity, allows for computing exact bounds in the output error of the aggregation function by knowing the error present in the input. An example of how the property of Lipschitz continuity of an aggregation function may be put to practical use in sustainability assessment is given next. Table 2 includes definitions for standard continuity and Lipschitz continuity for comparison and reference.

2.3. Example: Lipschitz Continuity and Error Estimation in the Arithmetic Mean

Error estimation and uncertainty quantification through the aggregation process may be approached by utilizing a variety of techniques. Certain aggregation functions have properties that allow one to provide

Table 1
Example aggregation functions.

Function name	Formula	Assumptions/notes
Arithmetic mean	$A(x) := \frac{1}{n}\sum_{i=1}^{n} x_i$	$A : \mathbb{I}^n \rightarrow \mathbb{I}, x : \in \mathbb{I}$
Weighted arithmetic mean	$A(x) := \sum_{i=1}^{n} w_i x_i$	$A : \mathbb{I}^n \rightarrow \mathbb{I}, x : \in \mathbb{I}(w_1, \ldots, w_n) \in [0, 1]^n \sum_{i=1}^{n} w_i = 1$
Ordered weighted average	[a] $A(x) := \sum_{i=1}^{n} w_i x_{(i)}$	$A : \mathbb{I}^n \rightarrow \mathbb{I}, x : \in \mathbb{I}$ $(w_1,..., w_n) \in [0, 1]^n \sum_{i=1}^{n} w_i = 1$
Geometric mean	$A(x) := (\prod_{i=1}^{n} x_i)^{1/n}$	$A : \mathbb{I}^n \rightarrow \mathbb{I}, x : \in \mathbb{I}$ [b] If $n > 1$ then $\mathbb{I} \subseteq (0, \infty)$
Weighted geometric mean	$A(x) := \prod_{i=1}^{n} x_i^{w_i}$	$A : \mathbb{I}^n \rightarrow \mathbb{I}, x : \in \mathbb{I}$ $(w_1,..., w_n) \in [0, 1]^n \sum_{i=1}^{n} w_i = 1$ If $n > 1$ then $\mathbb{I} \subseteq (0, \infty)$
Minimum	$A(x) := min\{x_1,..., x_n\}$ (or $OS_1(x) := x_{(1)}$)	Also written $\min(x) = \bigwedge_{i=1}^{n} x_i$ and OS_1 is the 1st order statistic
Maximum	$A(x) := max\{x_1,..., x_n\}$ (or $OS_n(x) := x_{(n)}$)	Also written $\max(x) = \bigvee_{i=1}^{n} x_i$ and OS_n is the nth order statistic

[a] $x_{(i)}$ represents the ith lowest coordinate of x, s.t. $x_{(1)} \leq \cdots \leq x_{(k)} \leq \cdots \leq x_{(n)}$.
[b] The geometric means are not aggregation functions on every domain, specifically, for $n > 1$ then $\mathbb{I}$ must satisfy $\mathbb{I} \subseteq (0, \infty)$.