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# Discrete-time optimal asset allocation under Higher-Order Hidden Markov Model

Dong-Mei Zhu[a], Jiejun Lu[b,*], Wai-Ki Ching[b], Tak-Kuen Siu[c]

[a] School of Management and Economics, Southeast University, Nanjing, China
[b] Advanced Modeling and Applied Computing Laboratory, Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong
[c] Department of Applied Finance and Actuarial Studies, Faculty of Business and Economics, Macquarie University, Macquarie University, Sydney, NSW 2109, Australia

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A B S T R A C T

This paper studies an optimal portfolio selection problem under a discrete-time Higher-Order Hidden Markov-Modulated Autoregressive (HO-HMMAR) model for price dynamics. By interpreting the hidden states of the modulating higher-order Markov chain as different states of an economic condition, the model discussed here may incorporate the long-term memory of economic states in modeling price dynamics and optimal asset allocation. The estimation of an estimation method based on Expectation-Maximization (EM) algorithm is used to estimate the model parameters with a view to reducing numerical redundancy. The asset allocation problem is then discussed in a market with complete information using the standard Bellman's principle and recursive formulas are derived. Numerical results reveal that the HO-HMMAR model may have a slightly better out-of-sample forecasting accuracy than the HMMAR model over a short horizon. The optimal portfolio strategies from the HO-HMMAR model outperform those from the HMMAR model without long-term memory in both real data and simulated data experiments.

## 1. Introduction

Asset allocation is one of the key research topics in financial economics. The seminal work of Markowitz (1952) pioneered the use of mathematical models to study the problem. Though being one of the classic models in modern finance, the Markowitz mean-variance portfolio selection model is a one-period model which may not be able to provide a flexible way to describe the situation where investors update investment decisions over time. A continuous-time asset allocation paradigm pioneered by Merton (1969, 1971) provides a solid theoretical basis to study how economic agents optimize their portfolios continuously over time. Instead of optimizing the mean-variance relationship, Merton considered the maximization of an expected utility. A closed-form expression for an optimal portfolio was derived under the assumptions of a Geometric Brownian motion for asset prices and a power utility. Though its theoretical soundness, the Merton asset allocation model does not seem to be very appealing from the econometric perspective. Indeed from the econometric perspective, it seems more convenient to consider a discrete-time optimal asset allocation model than its continuous-time counterpart since the estimation of the former may be easier than the latter given real financial data. Samuelson (1969) considered an optimal consumption-investment problem in a discrete-time modeling framework, where the objective was to maximize the expected consumption. Grauer and Hakansson (1982) considered an optimal asset allocation problem in a discrete-time financial model.

It seems quite well-known that, structural changes in economic conditions or political regimes indispensably result in a dramatic change in the financial market. A possible way to incorporate the changes in economic conditions or political regimes in strategic asset allocation is to consider time-varying optimal investment problems, such as Kim and Omberg (1996) and Campbell and Viceira (2002). In the past few decades, there has been some interest in the application of regime-switching models in economics, finance and econometrics. These models take account of the impact of transitions in macroeconomic conditions. Hamilton (1989, 1990) pioneered the applications of discrete-time, Markovian regime-switching models in economics and econometrics. As for asset allocation problems, one may consult Elliott and van der Hoek (1997), Ang and Bekaert (2002), Yin and Zhou (2003), Zhou and Yin (2004), Guidolin and Timmermann (2007), Yiu et al. (2010) and Song et al. (2012), etc. Elliott and Siu (2014) discussed a mean-variance utility optimization problem and its corresponding filtering problem in a fractional Gaussian process from which a hidden Markov chain is partially observed. Long-term memories in asset returns are incorporated by the fractional Gaussian noise process.

* Corresponding author.
*E-mail addresses:* dongmeizhu86@gmail.com (D.-M. Zhu), gwunlou@hotmail.com (J. Lu), wching@hku.hk (W.-K. Ching), ken.siu@mq.edu.au, ktksiu2005@gmail.com (T.-K. Siu).

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The above models employ Markov chains to describe the macroeconomic states. In particular, the Markov-switching autoregressive time series model for optimal asset allocation has been studied by Guidolin and Timmermann (2007). Nevertheless, financial time series are observed to have memories. Cajueiro and Tabak (2008) demonstrated the long-range dependence in LIBOR and US interest rates. McCarthy et al. (2009) investigated long memory in corporate bond yield spread. Higher-order hidden Markov model, where the next state in the Markov chain depends on the more prior states may provide a possible way to incorporate long memory in economic states. This idea has received increasingly attention in recent years. For example, Xi and Mamon (2011) analyzed an asset price model driven by a weak hidden Markov chain. Siu et al. (2009) examined the higher-order effect of a risky portfolio and investigated its risk measurement under higher-order hidden Markov model. Ching et al. (2007) explored pricing of exotic options with the Esscher transform under higher-order Markov assumption. Siu et al. (2005) applied a double higher-order hidden Markov chain model to estimate the hidden state of an economy using credit ratings and interest rates. For higher-order autoregressive hidden Markov models, more information from the past is included and theoretically this dependency could be extended to any value. The limitation is that estimation of parameters becomes more complicated as the enlargement of the states. For the first order hidden Markov model, such procedures were discussed in Rabiner (1989). However, when it comes to higher-order cases, few generalizations have been developed. Readers may refer to, for example, du Preez (1998), Ching et al. (2008), Ching et al. (2013) and Hadar and Messer (2009) for efficient computation ideas.

In this paper we discuss an optimal portfolio selection problem in a partially observed system. The price process of a risky asset is described by a discrete-time Higher-Order Hidden Markov-Modulated Autoregressive (HO-HMMAR) model. The rationale behind the model is to incorporate the impact and long-term dependence of changing hidden economic states on the price process. Based on the simple and elegant method proposed by Hadar and Messer (2009), we present an efficient approach to reduce numerical redundancy in optimally estimating parameters in higher-order models. Then we consider the optimal asset allocation problem by maximizing the expected power utility of terminal wealth. Using the standard Bellman's principle, we derive the corresponding recursive formulas for computing optimal trading strategies. Numerical experiments illustrate the implementation of the algorithm. Taking two risky assets, namely the S & P 500 index and the gold, into account, we make future predictions and compute the optimal strategies. Results suggest that the HO-HMMAR model may have a slightly better predicability than the 1-HMMAR model over a short forecasting horizon, say a one-day horizon. As for the wealth performances, the HO-HMMAR model outperforms the 1-HMMAR model in both real and simulated data. The results also reveal that the optimal trading strategies seem to be sensitive to the choices of the risk averse parameter. An advantage of the HO-HMMAR model seems to be that it reacts more quickly and moderately to switches in market regimes based on either a single-step expected power utility or recursive formulas.

The rest of the paper is structured as follows. The next section presents the Higher-Order Hidden Markov-Modulated Autoregressive (HO-HMMAR) model. The estimation method is then discussed in Section 3. In Section 4, we present the optimal asset allocation problem and derive the recursive formulas for computing the optimal portfolio strategy. Numerical experiments are presented in Section 5. Out-of-sample predictions and performances of optimal strategies under the HO-HMMAR models with different orders are discussed. Finally, concluding remarks are given in Section 6.

## 2. Discrete-time Higher-Order Hidden Markov-Modulated Autoregressive (HO-HMMAR) model

In this section, we present a discrete-time HO-HMMAR time series model for the price dynamics of several correlated risky assets. We assume that the hidden state sequence $\{X_t\}$ is a discrete-time, homogeneous hidden Markov chain of order $r$ defined on a complete probability space $(\Omega, F, \mathbb{P})$ with state space being a finite state set $\mathcal{N}$, i.e., a stochastic process that satisfies

$$P(X_t|\{X_l\}_{l\le t-1}) = P(X_t|\{X_l\}_{l=t-r}^{t-1}) \quad (1)$$

Here the transition probability $P(X_t|\{X_l\}_{l=t-r}^{t-1})$ is independent of current time $t$, and $X_t$ takes a value in $\mathcal{N}$. The states of the chain are interpreted as different states of the hidden economy.

For each $t \in T$, where $T$ is the set of non-negative integers, let $r(t)$ be the one-period risk-free interest rate of the money market account, where $r(t) > 0$. Suppose $r(t){:=}r_n\mathbf{1}_{\{X_t=n\}}$ where $r_n$ is interest rate when the hidden state of the economy is in the $n^{th}$ state. Suppose there is a market in which $(m+1)$ assets (or securities) are traded. One of the assets is a riskless bond whose evolution is governed by

$$B(t) = B(t-1)(1 + r(t)).$$

The prices of the $m$ assets are denoted by $P_1(t), P_2(t), \ldots, P_m(t)$ and the log returns of these assets from time $t-1$ to $t$ are represented as

$$S_i(t) = \ln(P_i(t)/P_i(t-1)) \quad i \in \{1, 2, \ldots, m\}.$$

The return processes $\{S_i(t)\}_{i=1}^m$ are assumed to follow the HO-HMMAR model:

$$S_i(t) = \mu_i(t) + \sum_{j=1}^{q_i(t)} \beta_i^j(t) S_i(t-j) + \sum_{j=1}^{m} \sigma_{ij}(t)\epsilon_j(t), \quad (2)$$

where $\{\epsilon_j(t)\}_{j=1}^m$ is a set of $m$ independent and identically distributed (i.i.d.) standard normal random variables. Here $\{q_j(t)\}_{j=1}^m$ is a set of autoregressive orders taking values in the set of non-negative integers. Specifically, $\mu_i(t), \beta_i^j(t)$ and $\sigma_{ik}(t)$ $(1 \le i, k \le m)$ are assumed to be functions of $X_t, X_{t-1}, \ldots, X_{t-s+1}$:

$$\begin{cases} \mu_i(t) := \mu_i^n \mathbf{1}_{\{X_t=n_t, X_{t-1}=n_{t-1}, \ldots, X_{t-s+1}=n_{t-s+1}\}}, \\ \beta_i^j(t) := \beta_i^{j^n} \mathbf{1}_{\{X_t=n_t, X_{t-1}=n_{t-1}, \ldots, X_{t-s+1}=n_{t-s+1}\}}, \\ \sigma_{ik}(t) := \sigma_{ik}^n \mathbf{1}_{\{X_t=n_t, X_{t-1}=n_{t-1}, \ldots, X_{t-s+1}=n_{t-s+1}\}}. \end{cases}$$

Here $n$ denotes the state ensemble of $X_t, X_{t-1}, \ldots, X_{t-s+1}$ and $1_E$ is the indicator function of an event $E$. Hence the observable return processes $\{S_i(t)\}_{i=1}^m$ are generated by a joint probability distribution associated with the previous $q_i(t)$ terms of return processes $\{S_i(t)\}_{i=1}^m$ and the previous $s$ states of the hidden chain $\{X_t\}$.

To illustrate our model, the following notations are required:

1. $|\mathcal{N}|^{r+1}$ transition probabilities,

$$a_{n_0,\ldots,n_r} = P(X_t = n_0|X_{t-1} = n_1, \ldots, X_{t-r} = n_r). \quad (3)$$

2. $|\mathcal{N}|^s$ observation probability distributions,

$$b_{n_0,\ldots,n_{s-1}}(S_t) = P(S_t|X_t = n_0, \ldots, X_{t-s+1} = n_{s-1}). \quad (4)$$

3. $|\mathcal{N}|^\nu$ initial state probabilities,

$$\pi_{n_1,\ldots,n_\nu} = P(X_0 = n_1, X_{-1} = n_2, \ldots, X_{1-n_\nu} = n_\nu). \quad (5)$$

Here $\nu = \max\{r, s\}$ and $n_0, n_1, \ldots, n_\nu \in \mathcal{N}$. We denote the set of all model parameters by $\lambda$, i.e.,