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Explicit solutions to dynamic portfolio choice problems: A continuous-time detour

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ABSTRACT

Recently, many academic researchers have implemented several numerical procedures to solve a dynamic portfolio choice problem especially in incomplete markets. The subsequent numerical results are sometimes significantly different from one paper to another. Thus, they have all advocated the accuracy of their methods. This paper contributes to the previous accuracy debate by showing how to obtain some accurate numerical results without numerical approximations. We use the dynamic programming approach in continuous-time, and illustrate the framework with one risky and one riskless asset. The framework is flexible enough to cover all the HARA class of utility functions. We derive explicit solutions with a stochastic *market price of risk* and with a stochastic *volatility*. 7 countries are considered in numerical illustrations.

1. Introduction

In a practical point of view, the mean–variance framework is widely used for portfolio choice problems. This is due to the explicit solution it provides. In a theoretical point of view, quadratic utility is convenient to derive a classical mean–variance portfolio (also called *myopic demand*). Markowitz (2014) summarizes some of the related findings, and he clarifies some common mistakes such as the normality of return distributions. Yao et al. (2014) restate mean–variance portfolio in continuous-time using the dynamic programming approach. This allows them to establish some properties such as the no interplay between the risk free asset and the risk free wealth. However, going back to the seminal works on portfolio choices with the dynamic programming approach, Samuelson (1969) and Merton (1969) respectively in discrete-time and in continuous-time, show that a logarithmic utility or even a CRRA utility with constant investment opportunities, lead to mean–variance portfolio. Otherwise, an investor who has a complex utility and who believes in return predictability, should have an inter-temporal hedging demand to hedge against adverse changes in investment opportunities (Merton, 1971, 1973). The lack of easy-to-use explicit solutions with a realistic assumption such as incomplete markets, makes this task difficult.

Starting from the contribution of Merton (1971), many results on dynamic portfolio optimization problems have been obtained. However, with a simple CRRA utility, it still appears to be difficult to provide accurate numerical results when there is predictability in asset returns,[1] i.e. when investment opportunities are time-varying. A large number of papers have proposed to use a VAR model to forecast returns and study their implications on long-term portfolio choice problems. As a result, the academic literature has followed two main lines. The first one relies on mathematical tools, and then establishes some theoretical explicit solutions (see Kim and Omberg, 1996; Liu, 2007 and references therein). Such solutions exist only in continuous-time. To provide accurate numerical values, investor must solve a quite complicated issue of time aggregation (Bergstrom, 1984; Campbell et al., 2004). The second line of research directly implements in discrete-time some challenging numerical methods. In fact, Barberis (2000) develops a discretization state space method that serves as a benchmark. Brandt et al. (2005); van Binsbergen and Brandt (2007); Garlappi and Skoulakis (2009) among others use some sophisticated backward induction techniques and evaluate the accuracy of their results by comparing them to the discretization state space benchmark. Nevertheless, in a recent paper, Cong and Oosterlee (2015) implement an improved version of the method of Brandt et al. (2005), and then, they compare

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[1] In this paper, the word return means a log or continuously compounded total return unless otherwise stated.

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the resulting numerical values to a benchmark based on the Fourier cosine series expansion. Using a theoretical model, we show how to analytically obtain similar numerical values.

Some approximation numerical procedures have appeared to be inconsistent. In fact, Detemple et al. (2005) find that Detemple et al. (2003) procedure is more accurate and more faster than that of Brandt et al. (2005). van Binsbergen and Brandt (2007) using regression procedure to approximate the expectation component of value function claim that the portfolio weight iteration (which was previously developed by Brandt et al., 2005) is more accurate than that of the value function. Garlappi and Skoulakis (2009) challenge this result by showing that certainty equivalent transformation of value function leads to much more accurate numerical results when the expectation of the value function is approximated by Gauss–Hermite quadrature with six nodes. Garlappi and Skoulakis (2011) provide a general discussion on approximation accuracy in discrete-time. However, all discrete numerical procedures approximate directly or indirectly a highly non-linear value function and cannot explicitly separate the so-called *hedging demand* from the so-called *myopic demand*. The continuous-time model considered in this paper does this separation and provides some very accurate numerical results since it is based on a well-documented explicit solution.

Boyle et al. (2008); Detemple et al. (2005); Detemple et al. (2003); Cvitanić et al. (2003) among others work in continuous-time. They perform some pure simulation techniques to derive optimal portfolio weights. In fact, these authors achieve a transformation of portfolio weight as a fraction of instantaneous standard deviation of wealth or obtain solutions under Malliavin calculus, and then they carry different kinds of Monte Carlo simulations to provide numerical values. Unfortunately, these prominent techniques are intractable with the assumption of incomplete markets. We propose a direct approach derived from analytical formula with a realistic assumption of incomplete markets for all the HARA (Hyperbolic Absolute Risk Aversion) preferences. We derive explicit solutions when the *market price of risk* (also called *Sharpe ratio*) defined the state variable, and when the stochastic *volatility* defined the state variable in the sense of Heston (1993). We make a detour to construct a bridge between continuous-time and discrete-time parameters. Actually, the literature on return predictability forcefully claims that the log dividend-price ratio predicts stock returns (Fama and French, 1988; Campbell and Shiller, 1988; Hodrick, 1992 or more recently Cochrane, 2008, 2011). Therefore, taking the predictability as given, we based our analysis on the log dividend–price ratio, and in empirical illustration we consider 7 countries[2]: Canada, France, Germany, Italy, Japan, UK and US.

Unlike Campbell et al. (2004), we deal with horizon which does not need to be necessarily infinite, and we define the continuous state variable as the *market price of risk* (also called *Sharpe ratio*) rather than the risk premium. We focus on the clear link between a continuous state variable (*market price of risk*/*volatility*) and a discrete state variable (*log dividend–price ratio*). This leads to some comprehensive expressions, which are very fast to be implemented. Campbell et al. (2004) work with an approximate analytical solution for an investor with an infinite horizon and recursive preferences. In this context, they provide evidence that there should exist minor discrepancies between results under discrete-time vs. continuous-time models. Accordingly, numerical results we derive from continuous-time are indirectly comparable to those of Garlappi and Skoulakis (2009). We show that, for large degrees of risk aversion and/or small horizons, when the state variable is close to its unconditional mean, the two numerical results are quite similar. Otherwise, results under our explicit solutions in continuous-time exhibit some discrepancies with Garlappi and Skoulakis (2009) when the risk aversion decreases and/or the time horizon increases. We argue that this is due to the large sensitivity of total demand to the continuous-time state variable (*Sharpe ratio*) or equivalently to the discrete-time state variable (*log dividend–price ratio*).

The paper is organized as follows: Section 2 presents the general investment opportunity sets, it treats 2 underlying applications; Section 3 studies some dynamic portfolio choice problems in continuous-time; Section 4 exposes the way we map a continuous-time investment opportunity set and a discrete-time one; Section 5 illustrates some numerical results with international data, and it tests the accuracy of our approach using the estimates of Brandt et al. (2005) for the purposes of comparison; Section 6 concludes.

2. Investment opportunity sets

Investment opportunities in continuous-time are generally described by the connection between returns, interest rate and *volatility*. If at least one of these three variables is stochastic, then the investment opportunities become stochastic. A state variable drives the investment opportunities as well as the price changes. Thus, a state variable evolves in a predictable way respecting a probability measure. In this section, we firstly propose a general discussion with a free state variable. Secondly, we study two interesting applications.

2.1. General stochastic investment opportunity sets

An investor should react to changes in investment opportunities Merton (1971, 1973). Let us consider the following general diffusions:

$$\mathrm{d}(P_t+D_t)/(P_t+D_t)=\mu_r(S_t,t)\mathrm{d}t+\sigma_r(S_t,t)\mathrm{d}B_t^r, \quad (1)$$

$$\mathrm{d}S_t=\mu_s(S_t,t)\mathrm{d}t+\sigma_s(S_t,t)\mathrm{d}B_t^s, \quad (2)$$

where the variables $(P+D)$ and S respectively denote the real stock price (including the dividend D) and the real state variable S. The random variables $\mathrm{d}B_t^r$ and $\mathrm{d}B_t^s$ are two standard Brownian motions. All stochastic processes constructing from Brownian motions are supposed to be progressively measurable on a filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$ where Ω is the state space, $\mathcal{F}$ is the σ-algebra indicating measurable events, $\mathbb{P}$ is the historical probability and the filtration is the augmented filtration generated by the Brownian motions. Parameters $\mu(\cdot)$ and $\sigma(\cdot)$ naturally denote the drift and diffusion rates related to each equation. They are supposed to satisfy Lipschitz conditions guaranteeing that each differential equation admits a unique strong solution. When there is no confusion, we will respectively denote $\mu_r(S_t,t)$, $\mu_s(S_t,t)$, $\sigma_r(S_t,t)$ and $\sigma_r(S_t,t)$ by μ_r, μ_s, σ_r and σ_s. The drift rate $\mu_r(\cdot)$ depends on the level of real interest rate r assumed to be constant. Thus, we suppose that there exists a risk free asset whose real price P^f evolves such that

$$\mathrm{d}P_t^f/P_t^f=r\,\mathrm{d}t. \quad (3)$$

Hence, in our model, the investment opportunities are stochastic if and only if, the state variable S is stochastic meaning that its expectation is time varying and its *volatility* is different from zero.

The two Brownian motions in Eqs. (1) and (2) are correlated such that $\mathrm{d}B_t^r\mathrm{d}B_t^s=\rho_s\mathrm{d}t$. The instantaneous correlation between shocks is given by $\rho_s\neq\pm 1$, meaning that markets are incomplete. Thus the number of state variables could not exceed the number of risky assets (Merton, 1973). For this reason, we do not explicitly take into account the dynamic of inflation which acts as an additional state variable. We directly consider real variables. Furthermore, empirical investigations reveal that the inflation risk is minor. The quarterly *volatility* of inflation is very low in 7 countries (see Table 1). It is about 3.83/200 for the US during the period 1970Q1–2015Q3. However, note that if markets are complete, the issue of limited state variables disappears since shocks would be the same. Honda and Kamimura (2011); Detemple et al.

[2] When there is no confusion, we will respectively call these countries CAN, FR, GER, ITA, JAP, UK and US.

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