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Reducible diffusions with time-varying transformations with application to short-term interest rates

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ABSTRACT

Reducible diffusions (RDs) are nonlinear transformations of analytically solvable Basic Diffusions (BDs). Hence, by construction RDs are analytically tractable and flexible diffusion processes. Existing literature on RDs has mostly focused on time-homogeneous transformations, which to a significant extent fail to explore the full potential of RDs from both theoretical and practical points of view. In this paper, we propose flexible and economically justifiable time variations to the transformations of RDs. Concentrating on the Constant Elasticity Variance (CEV) RDs, we consider nonlinear dynamics for our time-varying transformations with both deterministic and stochastic designs. Such time variations can greatly enhance the flexibility of RDs while maintaining sufficient tractability of the resulting models. In the meantime, our modeling approach enjoys the benefits of classical inferential techniques such as the Maximum Likelihood (ML). Our application to the UK and the US short-term interest rates suggests that from an empirical point of view time-varying transformations are highly relevant and statistically significant. We expect that the proposed models can describe more truthfully the dynamic time-varying behavior of economic and financial variables and potentially improve out-of-sample forecasts significantly.

1. Introduction

Since the seminal work of Merton (1973), continuous-time diffusion models have proved to be extremely useful in modeling financial and economic dynamics. They have been frequently applied to research in consumption, savings and investment problems, contingent claim pricing, asset return dynamics and so on. In particular, probably more models have been put forward to explain the behavior of short-term interest rates (short-rates) than for any other issue in finance (cf. Chan et al., 1992).

The basic dynamics for univariate continuous-time diffusion $\{Y_t, t \geq 0\}$ is described by the following Stochastic Differential Equation (SDE), also known as Ito's diffusion:

$$dY_t = \mu_Y(Y_t)dt + \sigma_Y(Y_t)dW_t \tag{1}$$

where $\mu_Y(y)$ and $\sigma_Y^2(y)$ are the instantaneous drift and diffusion functions respectively, and $\{W_t, t \geq 0\}$ is a standard Brownian motion. Parametric diffusions, which form the majority in the literature, assume that μ_Y and σ_Y^2 are known functions up to an unknown parameter vector ϕ, i.e. $\mu_Y(y) = \mu_Y(y, \phi)$ and $\sigma_Y^2(y) = \sigma_Y^2(y, \phi)$. Well known examples of parametric diffusions in finance include Merton (1973); Brennan and Schwartz (1979); Vasicek (1977); Cox (1975); Dothan (1978); Cox et al. (1980, 1985); Courtadon (1982); Constantinides and Ingersoll (1984); Constantinides (1992); Duffie and Kan (1996); Aït-Sahalia (1996b); Conley et al. (1997); Ahn and Gao (1999) (AG); Bu et al. (2011) and so on. Nonparametric and semiparametric approaches which deviate from the full parametric assumptions have also been proposed in the literature for their functional flexibility. Notable examples include Aït-Sahalia (1996a); Stanton (1997); Jiang and Knight (1997); and Kristensen (2010); and most recently Bu et al. (2014).

From an econometric point of view, parametric diffusions often provide a more intuitive and convenient way to specify the dynamics of the state variable. In the meantime, it is also convenient to apply classical inferential techniques such as Maximum Likelihood (ML) and Method of Moments as long as the likelihood function or certain moment functions can be evaluated effectively. In this regard, inference for nonparametric or semiparametric diffusions can be significantly more complicated and inefficient (cf. Kristensen, 2010). Thus, from a practical point of view, parametric diffusions are much more abundant and widely used than nonparametric or semiparametric diffusions in empirical applications.

Consequently, a great deal of effort has been spent searching for efficient ways to estimate parametric diffusions. ML is typically the method of choice for its proclaimed efficiency gain. Nevertheless,

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diffusion models are specified in continuous time, but empirical data are always sampled at discrete-time intervals. Little can be said about the implications of the dynamics in Eq. (1) for longer time intervals. In finance, Black and Scholes (1973); Vasicek (1977); Cox et al. (1985); and Ahn and Gao (1999) are the few rare cases where the discrete-time transition Probability Density Function (PDF) is known in closed form. However, substantial nonlinearity beyond the assumptions of these cases has been documented in the literature. In the context of short-rate modeling, Aït-Sahalia (1996b) for example concluded that the majority of existing parametric diffusion models were rejected by his data. This then became the motivation behind his well known Aït-Sahalia (1996b) general parametric specification.[1] Meanwhile, in a fully nonparametric setting, Stanton (1997) also observed strong nonlinearity in diffusion models for the short-rate series. The difficulty of almost all nonlinear diffusions is twofold. On one hand, they normally have no closed-form transition PDFs. Hence, considerable energy has been employed in developing various density approximation schemes. However, a price has to be paid for approximation errors and computational burden (cf. Durham and Gallant, 2002). On the other hand, some parameters of highly nonlinear models can sometimes be hard to identify from the data (cf. Elerian et al., 2001). Therefore, the problem of flexible modeling and efficient estimation of nonlinear continuous-time diffusions remains to be an important issue in practice.

In view of these difficulties, Bu et al. (2011) proposed a novel approach for modeling diffusions using Reducible Diffusions (RDs). RDs are defined by Kloeden and Platen (1992) as monotone transformations of analytically tractable Basic Diffusions (BDs) that have closed-form solutions. Since RDs are usually constructed by nonlinear transformations, they are potentially more flexible to capture complex dynamics of stochastic processes but at the same time possess desirable analytical tractability inherited from tractable BDs. Bu et al. (2011) considered two classes of RDs. The first class are diffusions transformed from the Ornstein–Uhlenbeck (OU) process (cf. Vasicek, 1977) and the second are transformations of the square-root (CIR) process (cf. Cox et al., 1985). Since the OU and the CIR processes have renowned analytical tractability, both OU-reducible diffusions (OU-RDs) and CIR-reducible diffusions (CIR-RDs) have similar degrees of tractability. In the context of short-rate modeling, they investigated RDs with Constant Elasticity Variance (CEV) diffusion specification, i.e. $\sigma_Y^2(y) = \sigma_0^2 y^{2\theta}$, which they named as OU-CEV and CIR–CEV RDs respectively. They showed that OU-CEV and CIR–CEV RDs are power functions of the OU and the CIR processes respectively and nest many known parametric models that have exact closed-form transition PDFs.

Modeling with nonlinear RDs has a number of advantages. Firstly, since RDs are transformations of BDs, additional (often substantial) flexibility can be achieved by specifying suitable nonlinear transformations. Secondly, the most important property of RDs is that their transition PDFs can be expressed in closed form via a transformation of the closed-form transition PDFs of BDs. Thus, the likelihood function for discretely observed samples can be evaluated and then optimized efficiently and standard likelihood-based inferential techniques can be used conveniently. Thirdly, the conditional Cumulative Distribution Functions (CDFs) of RDs, also known as the cumulative transition distributions, are also in closed form. This property makes CDF-based or quantile-based analyses very convenient. Such examples include value-at-risk analysis (Jorion, 2006), pricing options (Black and Scholes, 1973), conditional copula modeling (Patton, 2006a,b and 2009), and evaluating predictive densities (Diebold et al., 1998). Finally, the monotonicity of the transformations of RDs implies that crucial time series properties of discretely observed RDs such as stationarity, ergodicity and mixing are trivially implied from their BDs. See Doukhan (1994) and Forman and Sørensen (2014) for more detailed discussions.

While RDs potentially have many important advantages, the specifications suggested by Bu et al. (2011) are relatively restrictive compared to the vast literature on nonlinear stochastic modeling. It is quite unlikely that their time-homogeneous structure can be sufficient in describing various empirical dynamics except for only a few very special circumstances. Therefore, useful generalizations of this valuable framework and feasible extensions of existing specifications are extremely important from both theoretical and practical points of view.

The main contribution of this paper is to propose a number of flexible and easy-to-implement extensions to the specification of RDs and examine their empirical performance. Our objective is to generate sufficiently flexible transition densities on the basis of time-homogeneous RDs while maintaining sufficient tractability so that classic inferential techniques such as ML estimation can be easily implemented. The concept of conditional time variation in financial modeling finds its root in the pioneering work of Engle (1982) Autoregressive Conditional Heteroskedasticity (ARCH) specification for conditional variances. This insight was then generalized by Hansen (1994) in his general Autoregressive Conditional Density (ACD) framework. Hansen's suggestion is to select a distribution which depends upon a low-dimensional parameter vector, and then allow this parameter vector to vary as a function of the conditional variables. While Hansen's approach assumes that the conditioning set is perfectly adaptive (i.e. observable in the filtration of the process), there is a popular view that the dynamics of economic variables may depend on different states of the world or regimes. This is often referred to as state-dependent dynamic behavior. Depending on whether or not the state of the world at any given point in time is known with certainty in advance, the regime process can be either deterministic or stochastic. The latter case is particularly appealing, since it effectively creates a two factor stochastic process. See Chang et al. (2014) for the latest development in regime switching stochastic processes.

Although in theory time variations can be imposed on all elements of the parameter vector of the baseline model, in practice this is not always feasible. On one hand, imposing time variation on too many parameters tends to reduce the tractability of the model. On the other hand, it also reduces the interpretability and economic justification of the econometric model. Thus, in this paper we only allow the transformation parameter to be time-dependent. In other words, we effectively restrict our attention to RDs with time-varying transformations. In fact, one potential interpretation of RDs is that the BDs represent the fundamental risk factor and the empirically observed processes are transformed measures of this risk process. In this sense, by allowing the transformations to be time-dependent in our RDs, this interpretation may be further enhanced and enriched.

Since the philosophy behind our extensions is applicable to all parametric RDs, our exposition will focus on the CIR–CEV RD only due to its parsimony. Another reason of this choice is that the underlying BD (i.e. CIR) has a non-Gaussian transition PDF, which to some extent reflects the general need for deviation from the classic Gaussian framework. Moreover, the domain of the CIR–CEV RD can be more conveniently defined on the positive real line than the OU-CEV specification. This property is particularly appealing for financial modeling since the support of many financial variables (e.g. nominal interests) must be positive.

As we will see in Section 3, the transformation function of the CIR–CEV RD depends on a single parameter θ. We therefore propose a total of five distinct time variation schemes to allow θ to be time-varying by introducing dynamics of θ_t to the model. In Model 1, we specify θ_t as a deterministic function of the first lag of the state variable, i.e. $Y_{t-\Delta}$ where Δ is the fixed time interval. In Models 2 to 5, we introduce a regime switching mechanism. Specifically, Model 2 is a Self-Exited Threshold (SET) regime switching process where the threshold variable is taken as $Y_{t-\Delta}$. In order to allow regimes switching to be a continuous process, Model 3 is specified as a Logistic Smooth Transition (LST) process. Note that for Models 1 to 3, the time dependence of θ_t are

[1] The Aït-Sahalia (1996b) specification assumes that $\mu_Y(y) = \alpha_{-1}y^{-1} + \alpha_0 + \alpha_1 y + \alpha_2 y^2$ and $\sigma_Y^2(y) = \beta_0 + \beta_1 y + \beta_2 y^{\beta_3}$.

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