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Stochastic unit root processes: Maximum likelihood estimation, and new Lagrange multiplier and likelihood ratio tests☆

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ABSTRACT

We show in this study that the maximum likelihood estimators of stochastic unit root (STUR) processes are consistent and asymptotically normally distributed. We also present two new tests for STUR. We first propose a Lagrange multiplier test and show that it has a standard χ^2 distribution asymptotically. We also propose a likelihood ratio test and show that it has an asymptotic distribution of 50–50 mixture of χ^2 and a point mass at 0. As an empirical example, we test the existence of STUR in the Canadian real exchange rate and explore the implication of STUR on the validity of purchasing power parity.

1. Introduction

A variety of unit root tests have been developed over the years, and many economic variables are routinely found to have a unit root, or $I(1)$. See, for example, Haldrup and Jansson (2006) for a recent survey on unit root tests. Multivariate methods such as cointegration theory are based on the notion that the variables under study are nonstationary, or $I(1)$. However, the assumption that an autoregressive root is always equal to one seems to be unnecessarily restrictive. A simple and flexible generalization of unit root processes that does not deviate substantially from the current practice of $I(1)$ is a stochastic unit root (STUR) process, whose autoregressive root is equal to one only on average. There is a strong belief that STUR is prevalent among economic variables, see Granger (2000), and many economic variables that are believed to be $I(1)$ are already found to be better characterized as STUR.

STUR processes are introduced in Granger and Swanson (1997), Leybourne et al. (1996a, 1996b). The latter two papers also provide STUR testing procedures. The STUR tests have nonstandard asymptotic distributions, and their critical variables are tabulated in each paper. Taylor and van Dijk (2002) discuss the performance of the tests under various data-generating processes. See also Distaso (2008) for other tests for STUR. Estimation of STUR models is usually done with a Kalman filter; however, their asymptotic properties are not yet known. See also McCabe and Smith (1998) and McCabe and Tremayne (1995) for more discussions on STUR, and Francq et al. (2008) for an extension to bilinear-type STUR processes. Some properties of STUR are discussed in Yoon (2005). STUR processes are usually very difficult to distinguish from standard (fixed) unit root models by standard unit root tests.

In this study, utilizing the theoretical advances made in Ling (2004) and Klüppelberg et al. (2002), we show that the maximum likelihood (ML) estimators of the parameters of STUR processes are consistent and asymptotically normally distributed. We also propose a new Lagrange multiplier (LM) test for STUR, which follows a standard χ^2 distribution asymptotically under a normality assumption. We also consider a new likelihood ratio (LR) test for STUR, which has a 50–50 mixture asymptotic distribution of χ^2_1 and a point mass at 0.

It turns out that our STUR inference results follow easily from those discussed in Ling (2004) and Klüppelberg et al. (2002). In a remarkable paper, Ling (2004) studies the asymptotic distribution of ML estimators and proposes LM testing procedures for what he calls double-autoregressive (DAR) processes. STUR is a special case of his DAR processes. STUR is prevalent among economic variables, and hence, it would be fruitful to specify his general results to the case of STUR processes. Other estimation methods are also available in the literature for Ling's (2004) DAR models. For instance, Chan and Peng (2005) propose a weighted least absolute deviations method. See also Aue et al. (2006) for quasi-ML, Swaminathan and Naik-Nimbalkar (1997) for minimum distance, and Aue (2006) for least squares methods. The same is also true for our new LR test for STUR: it easily follows from the general results in Klüppelberg et al. (2002).

Finally, an empirical example is provided for the Canadian real exchange rate, for which standard unit root tests strongly indicate that the exchange rate is $I(1)$. We find evidence for STUR in the real exchange rate and discuss the implication of the existence of STUR on the validity of purchasing power parity.

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The remainder of this paper is organized as follows. In Section 2, STUR processes are discussed in the general framework of random coefficient autoregressive models. We derive the condition for the STUR to be strictly stationary and ergodic. In Section 3, quasi-ML estimators of the parameters of the STUR processes are presented, and their asymptotic distributions are provided. In Section 4, two new LM and LR tests for STUR are proposed, and their asymptotic distributions are given. In Section 5, Monte Carlo simulation results are presented for the performance of the ML estimators and the two new tests for STUR. In Section 6, an empirical example is discussed with the Canadian real exchange rate. We find strong evidence for STUR in preference to a unit root in the data. Finally, concluding remarks are given in Section 7.

2. Stochastic unit root processes

Consider the following simplistic time series model for y_t:

$$y_t = (\alpha + \eta_t) y_{t-1} + \varepsilon_t, \tag{1}$$

where α is a constant, $\eta_t \sim N(0, \sigma_\eta^2)$, and $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$ for $t = 1, \ldots, n$. n denotes the sample size. Additionally, η_t and ε_t are assumed to be independent of each other. The conditional variance of y_t is

$$Var(y_t | I_{t-1}) = \sigma_\varepsilon^2 + \sigma_\eta^2 y_{t-1}^2,$$

where I_t is the information set available at time t. Therefore, Eq. (1) exhibits level-dependent volatility.

Nicholls and Quinn (1982) call Eq. (1) a random coefficient autoregressive model of order one and show that Eq. (1) is weakly stationary if and only if

$$\alpha^2 + \sigma_\eta^2 < 1. \tag{2}$$

Eq. (1) becomes a random walk, or $I(1)$, when $\alpha = 1$ and $\sigma_\eta^2 = 0$:

$$y_t = y_{t-1} + \varepsilon_t. \tag{3}$$

Another particularly interesting class of models that belongs to Eq. (1) and does not deviate substantially from Eq. (3) is a (first-order) STUR process with $\alpha = 1$ and $\sigma_\eta^2 > 0$:

$$y_t = (1 + \eta_t) y_{t-1} + \varepsilon_t. \tag{4}$$

Eq. (4) reduces to a standard (fixed) unit root process or random walk Eq. (3) when $\sigma_\eta^2 = 0$. $E(\eta_t) = 0$, and hence, a STUR process Eq. (4) has an autoregressive root equal to one only on average. For STUR to be a plausible alternative to unit root processes, σ_η^2 should be "small". Other formulations of STUR processes are possible, but in this study, Eq. (4) will be employed for simplicity. Clearly, STUR process Eq. (4) does not satisfy the condition in Eq. (2) for weak stationarity. In fact, STUR cannot be transformed into a weakly stationary process by taking differences any number of times.

From Quinn (1982) and Bougerol and Picard (1992), the necessary and sufficient condition for the existence of a strictly stationary and ergodic solution of Eq. (1) is

$$E(\ln|\alpha + \eta_t|) < 0. \tag{5}$$

A strictly stationary and ergodic solution has the form of

$$y_t = \varepsilon_t + \sum_{j=1}^{\infty} \prod_{i=0}^{j-1} (\alpha + \eta_t) \varepsilon_{t-j}.$$

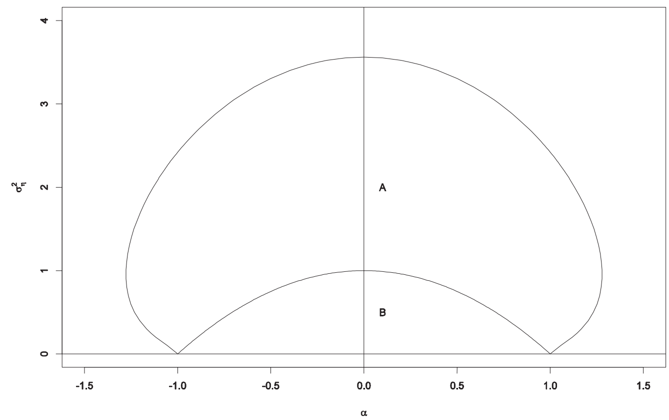

Fig. 1. Stationarity of random coefficient autoregressive processes, condition (6). Area A is from Eq. (6) for the strict stationarity of Eq. (1) and area B for weak stationarity in Eq. (2).

Under the normality assumptions and the independence of η_t and ε_t, Eq. (5) becomes

$$\ln\left(\sigma_\eta^2\right) < \gamma + \ln(2) - 2\int_0^1 \frac{1 - \exp\{-\upsilon(1 - w^2)\}}{1 - w^2} dw, \tag{6}$$

where γ is the Euler–Mascheroni constant: $\gamma = \lim_{m \to \infty} (\sum_{k=1}^{m} \frac{1}{k} - \ln(m)) \approx 0.5772$, and $\upsilon = \frac{\alpha^2}{2\sigma_\eta^2}$; see Theorem 2 in Wang (2003).

Area A in Fig. 1 shows the region of strict stationarity and ergodicity that satisfies condition Eq. (6).[1] Additionally, area B satisfies the condition for weak stationarity presented in Eq. (2). For instance, when $\alpha = 0$, for y_t in Eq. (1) to be strictly but not weakly stationary, σ_η^2 should be $1 \le \sigma_\eta^2 < \exp(\gamma + \ln 2) \approx 3.56214$. Note that y_t becomes the ARCH(1) model of Engle (1982) when $\alpha = 0$. Fig. 1 also shows that a random coefficient autoregressive process Eq. (1) could be strictly stationary and ergodic with $|\alpha| \ge 1$. For a STUR process with $\alpha = 1$, $0 < \sigma_\eta^2 < 2.42125$, approximately, for the strict stationarity condition Eq. (6) to be satisfied. It can be also shown that a STUR process is geometric ergodic under Eq. (5). In particular, it has a unique stationary distribution and satisfies the mixing condition with a geometric rate of convergence. The stationary distribution is continuous and symmetric. See Theorem 3 in Borkovec and Klüppelberg (2001), which contains additional results not mentioned here.

Thus, a STUR process is volatility-induced stationary under Eq. (5) in the sense of Conley et al. (1997). Many economic variables are routinely found to be $I(1)$ by standard unit root tests. For a STUR process to be a reasonable alternative to $I(1)$ processes, σ_η^2 should be "small." Fig. 2 shows some simulated STUR processes for various values of σ_η^2 with $n = 300$ and $\sigma_\varepsilon^2 = 1$. Random walk processes with the same $\{\varepsilon_t\}$ are also plotted in each panel for comparison. When σ_η^2 is small, it is very hard to distinguish between the two processes. As σ_η^2 increases, the STUR process becomes more erratic in that extreme values are more likely to occur.

Random coefficient autoregressive model Eq. (1) is known to be equivalent in distribution under Eq. (5) to the following autoregressive model with conditional heteroskedasticity:

$$y_t = \alpha y_{t-1} + \xi_t, \qquad \xi_t = \zeta_t \sqrt{\sigma_\varepsilon^2 + \sigma_\eta^2 y_{t-1}^2}, \tag{7}$$

where $\zeta_t \sim N(0, 1)$ and y_0 is independent of $\{\zeta_t : t \ge 1\}$. Ling (2004) calls Eq. (7) the first-order double-autoregressive model, or DAR(1). He also shows that the maximum likelihood estimators of $\{\alpha, \sigma_\varepsilon^2, \sigma_\eta^2\}$ are consistent and asymptotically normally distributed. Ling (2004) also proposes LM tests for the stationary DAR(1) model. In this study, we will

[1] Our Fig. 1 is the corrected version of Fig. 1 in Ling (2004).