Economic Modelling 51 (2015) 263–268

Contents lists available at ScienceDirect

Economic Modelling

journal homepage: www.elsevier.com/locate/ecmod

On the stochastic elasticity of variance diffusions☆

Jeong-Hoon Kim [a,*], Ji-Hun Yoon [b], Jungwoo Lee [a], Sun-Yong Choi [c]

[a] Department of Mathematics, Yonsei University, Seoul 120-749, Republic of Korea
[b] Department of Mathematics, Pusan National University, Pusan 609-735, Republic of Korea
[c] Department of Mathematics, Pohang University of Science and Technology, Pohang 790-784, Republic of Korea

ARTICLE INFO

Article history:
Accepted 13 August 2015
Available online xxxx

Keywords:
Stochastic elasticity of variance
Multiscale
Leverage effect
Implied volatility

ABSTRACT

The elasticity of variance of risky assets has been observed to be rapidly fluctuating around a level. The level itself slowly varies depending upon the corresponding economic situation at the time of consideration. In particular, it turns out to be extraordinary during the peak period of the 2007–2009 Global Financial Crisis. Based on the concept of stochastic elasticity of variance, this paper develops an asset price model in a multiscale form and applies it to the pricing of European options and verifies a significant improvement over the constant elasticity of variance model in terms of the geometric structure (skew or smirk) of implied volatility. Our result implies that a theoretical model based on the random elasticity can derive market's volatility forecast more accurately than the constant elasticity so that investors can employ a dynamic investment strategy reducing risk more effectively.

1. Introduction

One way of overcoming the drawback of the geometric Brownian motion assumption in the standard Black–Scholes model (Black and Scholes, 1973) is to assume that volatility of a risky asset depends on the underlying asset price itself so as to produce the skew of implied volatility of an option. The constant elasticity of variance (CEV) diffusion, which was initially introduced to finance by Cox (1975) and Cox and Ross (1976), is a renowned process belonging to this type of dynamics for risky assets. It has been popularly used by practitioners. One disadvantage in the CEV model (in general, local volatility models) is that the implied volatility might not move to the right direction as the underlying asset price increases as shown in Hagan et al. (2002). This faulty dynamics may yield a problem of instable hedging. There are also studies suggesting that the elasticity of variance is time varying instead of being a constant. For instance, Harvey (2001) used the technique of nonparametric density estimation to provide an empirical evidence on this time dependence and Ghysels et al. (1996) provide partial surveys on the time dependent volatility.

However, a recent study by Kim et al. (2014) on the S&P 500 index shows that it is rather randomly fluctuating than deterministic. This observation directly motivates a relaxation of the constant or time deterministic elasticity assumption leading to the so-called 'stochastic elasticity of variance' (SEV) model. Given stock price X_t, dynamics of this model can be expressed by the stochastic differential equation (SDE)

$$dX_t = rX_t dt + \sigma X_t^{\gamma_t} dW_t^x$$

under a risk-neutral measure, where r (interest rate) and σ (volatility coefficient) are positive constants, γ_t is a nonnegative stochastic process, and W_t^x is a standard Brownian motion. Here, we note that the formal definition of the elasticity of variance for the model is given by

$$\eta_t = 2(\gamma_t - 1).$$

The elasticity η_t controls directly the relationship between the volatility and the price of an underlying asset and thus it becomes a central role of the model. Of course, when $\eta_t = 0$, X_t becomes a geometric Brownian motion of the Black–Scholes model. If $\eta_t < 0$, then the model produces the so-called leverage effect, i.e., a negative correlation between returns and volatility, commonly observed in equity markets. See Campbell (1987), Breen et al. (1989), Glosten et al. (1993), and Brandt and Kang (2004) for example. Conversely, if $\eta_t > 0$, then it gives rise to the so-called inverse leverage effect, which means that volatility tends to increase with the level of underlying prices, often discovered in commodity markets (e.g., Emanuel and MacBeth (1982); Geman and Shih (2009)).

Even if the elasticity process η_t has been observed to be negative in stock markets, we find a very interesting phenomenon during a particular period of the 2007–2009 Global Financial Crisis. Namely, the mean level of η_t for the S&P 500 index became 'positive' in 2008. See Table 1

☆ The research of J.-H. Kim was supported by the National Research Foundation of Korea NRF-2013R1A1A2A10006693 and the research of S.-Y. Choi was supported by POSTECH Math BK21 Plus Organization.

* Corresponding author. Tel.: +82 2 21232600.
E-mail addresses: jhkim96@yonsei.ac.kr (J.-H. Kim), yssci99@pusan.ac.kr (J.-H. Yoon), isotope0@yonsei.ac.kr (J. Lee), lovesun@postech.ac.kr (S.-Y. Choi).

http://dx.doi.org/10.1016/j.econmod.2015.08.011

Table 1
Mean and standard deviation of η_t for the S&P 500 index.

Year	2005	2006	2007	2008	2009	2010	2011	2012	2013
Mean	−0.0680	−0.0893	−0.0318	0.0110	−0.1612	−0.1693	−0.0836	−0.1893	−0.1091
Standard deviation	0.1292	0.1421	0.1662	0.1762	0.1622	0.1754	0.1766	0.1524	0.1461

for yearly statistics of η_t. This switch of sign from negative to positive represents an 'anomaly' in the market during the peak period of the financial crisis clearly well in a unique way. Recall that on September 15, 2008, Lehman Brothers, which was the fourth-largest investment bank in the US, declared bankruptcy following the massive escape of clients, sharp losses in its stock price, and devaluation of its assets by credit rating agencies.

As noticed from Table 1, the average value of η_t is slowly varying with time but not necessarily constant zero as assumed in the paper by Kim et al. (2014). This motivates us to propose an extended model in which the elasticity η_t (or γ_t) is a non-zero constant plus a random process. To maximize analytic tractability for the SEV diffusion, the random part is assumed to be driven by a fast mean-reverting process while having a small amplitude. This type of model formulation allows us to use the multiscale asymptotic method of Fouque et al. (2011) which is based on averaging principle (homogenization) for stochastic equations with small parameters. By applying the multiscale hybrid model to the pricing of European options, we obtain a result improving the CEV model in view of implied volatility structure and yet the CEV option price constitutes a grounding price and subsequent correction terms are given by the Greeks (Delta, Gamma, Speed and etc) of the CEV price. Given the fact that the CEV model is widely used by practitioners in the financial industry, in particular, for modeling equities and commodities, our findings are within easy access of implementation for practitioners. Since the implied volatility derived from market option prices is commonly believed to be the market's volatility forecast, investors can make use of the proposed theoretical model to measure market's volatility forecast more accurately. So, our approach may provide an advanced 'dynamic' investment strategy reducing risk more effectively if the investors act to take maximum advantage of those random changes of leverage effect by recommending more accurate level of leveraged or inverse leveraged stock index funds.

The paper is organized as follows. Section 2 derives a multiscale partial differential equation for the option price based upon the SEV model. Section 3 is devoted to obtain an approximate option price using a Taylor expansion in one small parameter as well as a singular perturbation in the other small parameter. Section 4 discusses the geometry of implied volatility surface predicted by the SEV model by comparing it with the one given by the CEV model. Concluding remarks are given in Section 5.

2. A multiscale hybrid model

Let $\theta_t = 2\gamma_t$ for convenience and suppose that θ_t follows an Ornstein–Uhlenbeck process given by solution to the SDE

$$d\theta_t = \alpha(\mu - \theta_t)dt + \beta dW_t,$$

where α, β and μ are constants. To simulate the process θ_t, we use a discretized first-order autoregressive process (cf. Dixit and Pindyck (1994)) given by

$$\theta_t = e^{-\alpha\Delta t}\theta_{t-1} + \left(1 - \alpha e^{-\alpha\Delta t}\right)\mu + \beta\sqrt{\frac{\left(1 - e^{-2\alpha\Delta t}\right)}{2\alpha}}W_t,$$

where α, β and μ are calibrated using the maximum likelihood technique. Fig. 1 shows a comparison of the simulated process θ_t with the corresponding S&P 500 index historical data during 250 trading days from the 5th of April, 2010. Here, the construction of θ_t from historical data is obtained by

$$\theta_t = \frac{\log\sigma_t/\sigma}{\log S_t} + 1,$$

where σ_t is the realized volatility at time t and σ is the mean of the realized volatility over that period. We note particularly that $\alpha = 250$ in the simulation is a large number.

Table 1 shows the mean and standard deviation of the elasticity $\eta_t = \theta_t - 2$ for the S&P 500 index data for nine years from 2005 through 2013. Note that the average value of η_t is not zero. It is usually less than zero but it is bigger than zero exceptionally in 2008 which corresponds to the time period of culmination of the Global Financial Crisis. It is surprising to discover this inverse leverage effect even in an equity market. So, the usually observed negative correlation between returns and volatility in equity markets breaks down if financial situation becomes extremely risky. On the other hand, Fig. 1 shows the simulated θ_t and the empirical θ_t for the S&P 500 index, the KOSPI 200 index and the SSE composite index representing the US, South Korea and China market, respectively. They all show that the elasticity of variance is rapidly fluctuating around a certain level.

Based upon the results above with Fig. 1 and Table 1, we employ a multiscale framework developed by Fouque et al. (2011) to formulate the SEV diffusion X_t as follows. Using two small positive parameters ε and δ, a stochastic system for the price of an underlying risky asset is given by

$$dX_t = rX_t dt + \sigma X_t^{\frac{1}{2}\theta + \sqrt{\delta}f(Y_t)} dW_t^x, dY_t = \left(\frac{1}{\epsilon}(m - Y_t) - \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\Lambda\right)dt + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\, dW_t^y \tag{1}$$

under a risk-neutral probability measure, where r, σ, θ, m and v are positive constants. The inequality $\frac{-\theta}{2\sqrt{\delta}} \leq f \leq \frac{2-\theta}{2\sqrt{\delta}}$ and the absorbing boundary condition at $t = 0$ are required to admit a (unique) solution of system (1). The Brownian motions W_t^x and W_t^y are assumed to be correlated with correlation coefficient ρ. The market price of elasticity risk Λ is assumed to be independent of x and y for simplicity. Y_t is a Gaussian process having an invariant distribution given by $N(m, \nu^2)$. Notation $\langle \cdot \rangle$ is going to be used for expectation with respect to this probability distribution. It is worth noting that a family of the SEV diffusions X_t^θ is considered in this paper although the index θ is omitted throughout.

From the well-known Feynman–Kac formula (see, for example, Oksendal (2003)), the price of a European option with payoff h, defined by $P^{\epsilon,\delta}(t, x, y) = E^*[e^{-r(T-t)}h(X_T)|X_t = x, Y_t = y]$, satisfies

$$\mathcal{L}^{\epsilon,\delta}P^{\epsilon,\delta}(t,x,y) = 0,\ t < T,\ \ \mathcal{L}^{\epsilon,\delta} := \partial_t + \mathcal{L}_{X,Y}^{\epsilon,\delta} - r, P^{\epsilon,\delta}(T,x,y) = h(x), \tag{2}$$

where $\mathcal{L}_{X,Y}^{\epsilon,\delta}$ is the infinitesimal generator of the joint diffusion process (X_t, Y_t) and it is given by

$$\mathcal{L}_{X,Y}^{\epsilon,\delta} = \frac{1}{2}\sigma^2 x^{\theta + 2f(y)\sqrt{\delta}}\partial_{xx}^2 + rx\partial_x + \frac{1}{\sqrt{\epsilon}}\left(\rho\nu\sqrt{2}\sigma x^{\frac{\theta}{2} + f(y)\sqrt{\delta}}\partial_{xy}^2 - \nu\sqrt{2}\Lambda\partial_y\right) + \frac{1}{\epsilon}\left(\nu^2\partial_{yy}^2 + (m - y)\partial_y\right). \tag{3}$$