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A portfolio-invariant capital allocation scheme penalizing concentration risk

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ABSTRACT

In the internal ratings–based (IRB) approach under the revised Basel II, a well-suited risk capital scheme should meet the desirable property of portfolio-invariance, without which a sector's marginal capital contribution can be different when the composition of other sectors in the portfolio varies. However, an allocation scheme of the risk measure *VaR* can be portfolio-invariant only under the asymptotically single-risk factor (ASRF) framework, which understates the economic capital of a highly concentrated portfolio in a multi-risk factor environment. This study proposes a portfolio-invariant capital allocation scheme of *VaR* of an asymptotically fine-grained portfolio in a multi-risk factor environment. To penalize the concentration risk, the strategy for the proposed capital allocation scheme is to estimate the second-order polynomial that approximates the risk measure *VaR* using the response surface methodology (RSM). Comparisons are made between the proposed capital allocation scheme to three other capital allocation schemes including the approximated Euler capital allocation scheme, and the schemes based on the approximated single-risk factor approach and the diversification factor approach, respectively. The results indicate that the proposed RSM allocation scheme is the only scheme among the four that is portfolio-invariant and penalizes the sectors with concentrated exposures.

1. Introduction

The capital allocation scheme for the risk measure value-at-risk (*VaR*), which has been the dominant risk measure with all the desirable characteristics, is an important tool for risk identification and was written by the Basel II into the regulation of the finance industry (Basel Committee on Banking Supervision, 2004, 2006a, 2006b). To guarantee stability in business operations at the transaction level in order to make the internal ratings–based (IRB) approach applicable to a wider range of countries and institutions, an essential requirement of a capital allocation scheme by the Basel II is portfolio-invariance that ensures similar sub-portfolios are treated symmetrically (Basel Committee on Banking Supervision, 2006a, 2006b; Tarashev and Zhu, 2008). In other words, the capital required for any sub-portfolio will depend on its own risk characteristics and should be independent of the composition of the portfolio it is added to. In a banking portfolio, for example, the capital allocation among borrowers should depend only on individual borrower's credit rating, loan type, industrial sector, etc., instead of the compositions of the portfolio (see Gordy, 2003). The requirement of portfolio invariance has been deemed vital: it is convenient for additivity and helps to minimize the computational costs of implementing the IRB approach in the revised Basel II (Basel Committee on Banking Supervision, 2005).

The desire for portfolio invariance can never be overemphasized, without which it is even possible to have legal consequences. Nevertheless, Gordy (2003, 2004) showed that the capital allocation of *VaR* can meet the portfolio-invariant axiom only in an asymptotically fine-grained portfolio under a single systematic-risk factor environment. For this reason, the Basel Committee on Banking Supervision (2006a, 2006b) had adopted the asymptotic single systematic-risk factor (ASRF) model proposed by Gordy (2003) as the framework of the IRB approach under the revised Basel II. In Tsai and Chen (2011), a general formula for the capital allocation scheme under a single systematic-risk factor environment with any type of distribution of the asset returns (losses) is derived.

On the other hand, the assumption of single systematic-risk factor is over-simplified. As noted by Gordy (2003), a single factor cannot capture the diversification effect existing within a portfolio, such as the country-specific risk that borrowers of different countries are exposed to different macroeconomic risk factors (Castro, 2013; Iglesias, 2015), and the industry-specific risk that different industries can experience different cycles and thus different systematic risk factors (Castro, 2013; Iglesias, 2015). When a portfolio is exposed to different systematic risk factors, a capital allocation scheme based on the single systematic-risk factor model will significantly understate the capital needed to support a sector. The extent to which a single systematic-risk factor model understates the economic capital depends on the degree of the concentration of the exposures to specific sectors, as well as the degree of correlation among different systematic risk factors. In other words, the capital allocation scheme based on the single systematic-risk factor model cannot capture the concentration risk within a multi-systematic-risk factor environment, which has been the general critique of the Basel II (Basel Committee on Banking

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Supervision, 2006a, 2006b; Jarrow, 2007). As the concentration risk plays a critical role in banking failures (Basel Committee on Banking Supervision, 2004, 2006a, 2006b), adjustments to the ASRF model are developed so the diversification effect arising from a multi-risk factor setting can be accounted for. The adjustments include the approximated single systematic-risk factor approach (AP) by Pykhtin (2004), and the diversification factor (DF) approach by Garcia Cespedes et al. (2006).

This study proposes a capital allocation scheme for an asymptotically fine-grained portfolio in a multi-systematic-risk factor environment. The proposed capital allocation scheme is portfolio-invariant in a sense that sector k's marginal capital contribution will not be affected by the exposure weights of other sectors in the portfolio. At the same time, sector k's marginal capital contribution is increasing in its exposure weight to penalize the concentrated exposure. The strategy to obtain the proposed capital allocation scheme is to use the response surface modeling (RSM) approach to fit the risk measure *VaR*, a non-linear function in the exposure weights, by a second-order canonical polynomial over the simplex region Ω where the exposure weights are defined.

To illustrate a multi-systematic-risk factor model fits better than a single-risk factor model, the financial indices, i.e., the asset's return rates (ROAs), of a total 118 companies on the list of Taiwan Stock Exchange from January 2009 to December 2013 are analyzed. A numerical example is given to illustrate the proposed capital allocation scheme and to compare it with the other three capital allocation schemes, including the Euler allocation scheme, the schemes obtained from the approximated single-risk factor approach (AP) and the diversification factor (DF) approach, respectively. It is demonstrated that the proposed capital allocation scheme is the only scheme that satisfies the portfolio-invariance defined as above, and at the same time, accounts for the concentration risk as compared with the other three capital allocation schemes.

The paper is organized as follows. The Euler capital allocation scheme of a coherent risk measure is reviewed in Section 2. In Section 3, the Euler capital allocation scheme of the risk measure *VaR* of an asymptotically fine-grained portfolio in a multi-risk factor environment is derived. It is shown that the Euler capital allocation scheme is portfolio-dependent in a multi-systematic-risk factor environment. In Section 4, a portfolio-invariant capital allocation scheme penalizing the concentration risk is proposed. In Section 5, the proposed capital allocation scheme is compared with three other capital allocation schemes in a seven-systematic-risk factor environment, including the approximated Euler allocation scheme, the schemes obtained from the approximated single-risk factor approach (AP) by Pykhtin (2004) and the diversification factor (DF) approach by Garcia Cespedes et al. (2006). Section 6 concludes.

2. Euler capital allocation scheme of coherent risk measure

In the literature, a variety of risk capital allocation schemes are given (Csóka et al., 2009; Denault, 2001; Hallerbach, 2002; Homburg and Scherpereel, 2008; Jorion, 2007; Kalkbrener, 2005; Kalkbrener et al., 2004; Koyluoglu and Stoker, 2002; Kurth and Tasche, 2003; Martin et al., 2001; Mausser and Rosen, 2008; Rosen and Saunders, 2010; Tasche, 2004, 2006, 2008). Among these, the Euler allocation scheme a coherent risk measure ρ is

$$\rho(L(\boldsymbol{e})) = \sum_{k=1}^{K} e_k[\partial\rho(L(\boldsymbol{e}))/\partial e_k] \tag{1}$$

where

$$L(\boldsymbol{e}) = \sum_{k=1}^{K} e_k R_k,$$

$R_1, \ldots, R_K$ are the returns (losses) per unit of exposure of K assets,
Ξ is the vector space spanned by $\{R_1, \ldots, R_K\}$, and
$\boldsymbol{e} = (e_1, \ldots, e_K) \in R^{+K}$ are the exposure weights in the K assets.

By Artzner et al. (1999), a coherent risk measure ρ is continuously differentiable satisfying:

$$\text{Translation invariance: for all } L\in\Xi \text{ and a constant } c, \rho(L+c) = \rho(L) + c \tag{2a}$$

$$\text{Homogeneous of degree one: for all } L\in\Xi \text{ and } \lambda>0, \rho(\lambda L) = \lambda\rho(L) \tag{2b}$$

$$\text{Sub-additive: for all } L_1\in\Xi \text{ and } L_2\in\Xi, \rho(L_1+L_2)\le\rho(L_1)+\rho(L_2) \tag{2c}$$

$$\text{Monotonicity: for all } L_1\in\Xi \text{ and } L_2\in\Xi \text{ with } L_1\le L_2, \rho(L_2)\le\rho(L_1) \tag{2d}$$

The Euler allocation scheme has received much attention due to the fact that it is the only capital allocation that meets the linear aggregation and diversification axioms of Kalkbrener (2005) or “no undercut” by Denault (2001) when the risk measure is sub-additive and homogeneous of degree one. If the risk measure is continuously differentiable, the Euler allocation scheme meets the continuity axiom of Kalkbrener (2005) and is the only risk capital allocation scheme that is RORAC compatible by Tasche (2008). In addition, if the risk measure is translation invariant, the Euler allocation scheme meets the riskless allocation axiom of Denault (2001).

For the risk measure *VaR*, if the distribution of the asset returns (losses) exhibit multivariate regular variation with finite first moment and has a positive density function that admits second-order moments, then *VaR* is a coherent risk measure (Daníelsson et al., 2013; Gourieroux et al., 2000). Thus, *VaR* can be decomposed by the Euler allocation scheme and meets the aforementioned capital allocation axioms. However, the Euler allocation scheme of *VaR* does not satisfy the portfolio-invariant axiom as defined by Denault (2001): a risk capital allocation scheme is portfolio-invariant if any two sub-portfolios i and j ($i \neq j$) with the same risk characteristics when joining with the remaining sub-portfolio not containing i and j, one has:

$$\rho\left(\sum_{k\neq i,k\neq j} e_k R_k + eR_i\right) - \rho\left(\sum_{k\neq i,k\neq j} e_k R_k\right) = \rho\left(\sum_{k\neq i,k\neq j} e_k R_k + eR_j\right) - \rho\left(\sum_{k\neq i,k\neq j} e_k R_k\right) \tag{3}$$

for any $\boldsymbol{e} = (e_1, \ldots, e_K)\in R^{+K}$. That is, i and j have the same marginal capital contribution per dollar of exposure. In Buch and Dorfleitner (2008), a necessary and sufficient condition for (1) to be portfolio-invariant is a risk measure ρ satisfying (2a)–(2c) and linear in the exposure sizes $\boldsymbol{e} = (e_1, \ldots, e_K)\in R^{+K}$. In the following section, the Euler allocation scheme of the risk measure *VaR* of an asymptotically fine-grained portfolio is given. It is shown that the Euler allocation scheme does not meet the portfolio-invariant axiom when the portfolio is exposed to multiple risk factors.

3. Euler allocation scheme of *VaR* in a multi-systematic-risk factor environment

Suppose each borrower in a portfolio can be assigned to a specific sector among $K > 1$ sectors, and each sector has n borrowers and is affected by one of the K systematic risk factors $Z_1, \ldots, Z_K$. In the Merton-type factor model (Merton, 1974; Vasicek, 2002), the K systematic risk factors $Z_1, \ldots, Z_K$ are correlated through a macro systemic risk factor Z_0 in the form

$$Z_k = \beta_k Z_0 + \sqrt{1-\beta_k^2}\eta_k \tag{4}$$

where $0 \le \beta_k \le 1$ is the correlation parameter between Z_k and Z_0; the macro systemic risk factor Z_0 and the idiosyncratic risk factors $\eta_1, \ldots, \eta_K$ are all standardized normally distributed and are mutually independent. Eq. (4) implies $E(Z_k) = 0$ and variance $\mathrm{Var}(Z_k) = 1$ for $1 \le k \le K$; in addition, the covariance $\mathrm{Cov}(Z_k, Z_j) = \beta_k\beta_j$, $1 \le k \ne j \le K$. For borrower j in