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Optimal stopping time with stochastic volatility

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ABSTRACT

This paper demonstrates how to convert a path-dependent optimal stopping time problem into a path-independent problem using a transformation analysis method. We test this method to deal with several problems, especially those in stochastic volatility environments. We introduce stochastic state variables into volatility dynamics and analyse the influence of state-variable volatile characters on investment stopping boundaries. For arbitrary coefficient circumstances, we set up a Riccati equation that satisfies the transformation. For circumstances involving Heston stochastic-volatility, we propose an analytical solution. This paper extends research on the optimal investment stopping issue to a stochastic investment opportunity environment. Our proposed method can enhance the ability of optimal investment stopping theory to describe the real capital market.

1. Introduction

Option pricing theory (Black and Scholes, 1973; Merton, 1973) helps us to successfully understand and tackle many financial problems that we could not previously solve. For example, we can treat the optimal stopping time problem in terms of an exercise-time issue for an American-style option. By such an analogy, we can effectively capture the main characteristics of investment decisions, such as uncertainty of future cash flows or flexibility in timing.

We should consider many factors in the real option pricing process, including market friction, attitude toward risk or the wider investment environment. The classic studies such as that of McDonald and Siegel (1986) are all based on perfect market models and risk-neutral investors. These models assume that investment payoffs can be duplicated by tradable assets. However, such assumptions may be unrealistic. Recently, many studies have focused on how to relax these assumptions. For example, Grenadier and Wang (2005) take agency costs into account. Miao and Wang (2007) relax the perfect-market assumption in an environment of imperfect hedges and fuzzy risk aversion. Miao and Wang (2007) extend the standard real options approach to analyse the implications of uninsurable idiosyncratic risk for the investment-timing aspect of entrepreneurial activities. These authors use a utility-maximisation framework in which an agent chooses his consumption and portfolio allocations and then undertakes an irreversible investment.

However, there is another important unrealistic assumption that has been made in previous studies. Almost all of these studies have assumed a deterministic investment environment. The typical dynamics of asset pricing involve a geometric Brownian motion, which implies constant drift and diffusion coefficients. However, it is widely recognised that in reality the volatility of a financial time series involves not only variations in time, but also in clustering. In fact, both the growth rate and the volatility of asset returns are stochastically time-varying, as Keim and Stambaugh (1986), Campbell and Shiller (1988) or Fama and French (1989) have indicated concerning the domestic market, and as Bekaert and Hodrick (1993) or Ferson and Harvey (1993) have indicated concerning the international market.

In the past several years, several studies on this topic have dealt with the applications of real options in investment procedure, such as the studies by Wong (2010), Wong and Yi (2013), Nishide and Nomi (2009), Liang et al. (2014) or Shibata and Nishihara (2012). Wong (2010) and Wong and Yi (2013) both examine the effect of irreversibility on investment. Nishide and Nomi (2009) construct a real options model in which a regime change is expected at a pre-determined future time. They then examine the effects of regime uncertainty on a firm's strategic investment decisions. These researchers show that just before the time of a regime change, firms should act as if the worst-case scenario was about to happen, even if an improved state is highly possible. Liang et al. (2014) use the Martingale method and incorporate variation inequality to study the optimal investment and consumption strategies for a retired individual who has the opportunity to choose a discretionary stopping time to purchase an annuity. Shibata and Nishihara (2012) examine the optimal investment-timing decision problem of a firm that is subject to a debt-financing capacity constraint. These authors find that the investment thresholds have a U-shaped relation

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with the debt capacity constraint. Although the financing constraint distorts investment timing, this constraint may also make the constrained firm more willing than a non-constrained firm to overinvest.

To sum up, the previous studies on the optimal stopping time for investments do not take return variability into account. The theoretical analyses on stochastic asset returns mainly focus on portfolio selection. Merton (1973) uses a stochastic state variable model in a portfolio-selection framework, and points out that at least one variable can affect investment opportunity, namely, interest rate. Many researchers have followed Merton's approach, such as Campbell and Viceira (2002) or Brennan and Xia (2000, 2001, 2002).

In fact, stochastic interest rates, stochastic growth rates and stochastic volatility can all induce path-dependent asset pricing problems. Our paper extends optimal investment theory to circumstances with stochastic volatility characteristics. We assume that the process of asset value growth is dependent on stochastic state variables. This assumption differs from that of Miao and Wang (2007). We work out an explicit solution when the exercising cost is zero. This solution can be used to analyse the effects of stochastic return on optimal investment-timing selection. Overall, we contribute to the literature in the following three ways.

First, we transform a path-dependent stopping time problem into a path-independent problem, and do so in a manner that applies under very general conditions. Therefore, we can directly use the existing conclusions of the optimal stopping time literature. Generally, it is easier to verify the conditions of variational inequality in a path-independent stopping time problem, as the equation involved is easier to solve. The application of this method is not limited to the field of optimal investment, but also applies more generally to other frameworks for optimal stopping.

Second, for cases in which the dynamics of capital growth are homogeneous, we change a two-dimensional optimal stopping time problem into a single-dimension problem based on an assumption of zero exercising cost. We also work out explicit solutions for some special cases, such as cases involving Heston stochastic volatility or Vasicek stochastic volatility.

Finally, our method assumes an incomplete market with CRRA investors and a scale-invariant asset value growth process. Our method can be used in cases of expected return and imperfectly correlated volatility. The proposed approach allows investors to use more information sources, such as the implied volatility inferred from derivative prices in the mature derivatives market.

This paper is organised as follows. The general theoretical framework and basic assumptions are set forth in Section 2. In Section 3, we deal with solutions under circumstances of homogeneity. In Section 4, we put forward solutions in more general cases. Specifically, we transform a path-dependent optimal selling time problem into a path-independent problem by using a transformation analysis. Applications to stochastic volatility and to risk-averse investors are given in Section 5. Section 6 concludes the paper.

2. Model and assumption

We denote (Ω, F, P) as a complete probability space, and (dz_x, dz_v) as a two-dimensional Brownian motion on (Ω, F, P). The correlation coefficient between these factors is ρdt. The function F_t represents a filtration of information generated by (dz_x, dz_v), which is used to measure the available information at time t. The dynamics of an asset value whose risk cannot be fully hedged are specified as follows:

$$\frac{dV}{V} = \mu_v(X,t)dt + \sigma_v(X,t)dz_v \tag{1}$$

where, $\mu_v(X, t)$ and $\sigma_v(X, t)$ are the expected return and the volatility functions, respectively. Dixit and Pindyck (1994) and Alvarez and Koskela (2004) have assumed a geometric Brownian motion return process. Miao and Wang (2007) and Brock et al. (1988) have assumed an arithmetic Brownian motion capital growth process.

Unlike these previous studies, our model allows the coefficient of the capital value process to depend on stochastic state variables. We denote X as a state variable, which is subject to

$$dX = \mu_x(X)dt + \sigma_x(X)dz_x. \tag{2}$$

There are at least two ways that the state variable affects the asset value, capital growth and terminal payoff functions. The investors' payoff is state-dependent. Let $P(\tau)$ $(\tau \geq 0)$ represent the value before selling. On expiration day, the asset's value is described as

$$P(0) = F(X,V). \tag{3}$$

This equation corresponds to the free boundary of the optimal stopping time problem. We assume that there is no fixed cost when executing the selling action.

The dynamic process of risk-free return is

$$\frac{dB}{B} = r(X,t)dt. \tag{4}$$

It should be noted that the market is incomplete, and investors cannot hedge risk by continuous trading (Miao and Wang, 2007). As a result, the Black–Scholes option pricing model cannot be directly applied. The expected return in Eq. (1) may not be equal to the risk-free interest rate. Our method can be thought of as a utility function pricing method as proposed by Henderson (2002).

Given the Markovian property, the asset value at time τ can be denoted as $P(X, V, \tau)$. Risk-neutral investors make their decisions based on the following equation:

$$P(X,V,\tau) \triangleq \sup_\tau E\left\{ \exp\left[-\int_0^\tau r(s)ds\right] F(X,V)\right\}. \tag{5}$$

The investors' utility function is described as $U' > 0 > U''$, which satisfies the following equation:

$$P(X,V,\tau) \triangleq \sup_\tau E\left[e^{-\beta\tau} U(F(X,V))\right]. \tag{6}$$

We can choose Eq. (5) or Eq. (6), according to differing situations.

3. Basic theory and solution under homogeneity

According to Friedman (1988), the optimal stopping problem can be expressed as a free-boundary problem, and the conditions of variational inequality are equivalent to those of the optimal stopping time. Therefore, the problem can be solved by either a variational inequality or a linear complementary function. Elliott and Ockendon (1982) and Kinderlehrer and Stampacchia (1980) both prove that the linear complementary condition is equivalent to the variational inequality condition.

Let $G(X, V)$ be the dividend function, and A be a differential operator. For the arbitrary normative function P,

$$\begin{aligned}\mathcal{A}(P) \triangleq & \frac{1}{2}P_{XX}\sigma_X^2(X) + P_X\mu_X(X) + P_t + G(X,V) - r(X)P \\ & + \frac{1}{2}P_{VV}V^2\sigma_V^2(X) + P_V V\mu_V(X) + P_{VX}V\sigma_X(X)\sigma_V(X)\rho\end{aligned}. \tag{7}$$

The subscripts of function P denote partial derivatives. According to Bensoussan (1982), a sufficient condition of the optimal stopping problem is given by the following lemma.