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Upper and lower bounds for convex value functions of derivative contracts

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ABSTRACT

The aim of this paper is to compute upper and lower bounds for convex value functions of derivative contracts. Laprise et al. (2006) compute bounds for American-style vanilla options by selected portfolios of call options. We provide an alternative interpretation of their numerical procedure as a stochastic dynamic program for which the Bellman value function is approximated by selected piecewise linear interpolations at each decision date. The stochastic dynamic program does not (directly) depend on portfolios of call options, but rather on a key ingredient: some transition parameters of the underlying asset. More in line with the literature on dynamic programming, our procedure is contract free and is well designed to accommodate all one-dimensional convex value functions of derivative contracts. In support of this, we revisit the numerical investigation of Laprise et al. (2006) and enlarge their findings to include options embedded in bonds under affine term-structure models of interest rates.

1. Introduction

Models for financial derivatives are useful for practitioners since they provide fair values and sensitivities that may be used as guides for trading. Several derivative contracts cannot be valued in a closed form and have to be approximated in some way. Examples include options with early exercise opportunities. Several valuing procedures have been proposed in the literature. They assume a discretization of the state space and build on the property that the approximate value converges to the true value for finer and finer discretizations. In general, for a given discretization, the approximation error is unknown, which is a major disadvantage from a practical point of view. Enveloping the value function to be computed allows one to provide an upper bound for the approximation error.

Laprise et al. (2006) compute upper and lower bounds for American-style vanilla options using selected portfolios of call options in multiplicative models. Instead of building on portfolios of call options to value derivatives, we design an equivalent procedure, based on stochastic dynamic programming (SDP), for which the Bellman value function is approximated by selected piecewise linear interpolations at each decision date. Our construction has four main advantages with respect to (Laprise et al., 2006):

1. SDP belongs to a well-known family of numerical procedures, while Laprise et al. (2006) is an ad-hoc procedure. Thus, in using SDP, one benefits from the accumulated knowledge in the field of dynamic programming. See for example Bertsekas (1995).
2. SDP separates the evaluation problem into two parts: the dynamics of the underlying asset, captured by some transition parameters, and the form of the derivative to be priced, captured by a convex function, while Laprise et al. (2006) mix between the two parts.
3. SDP accommodates all convex value functions of derivative contracts, while Laprise et al. (2006) consider only on vanilla options under the geometric Brownian motion.
4. SDP can be extended for pricing convex derivatives in high-dimensional state spaces, while the algorithm of Laprise et al. (2006), as it is designed, cannot.

An extensive literature on approximation methods for valuing derivatives is readily available. Commonly used methods are based on:

1. Quasi-analytic approaches (Barone-Adesi and Whaley, 1987; Bunch and Johnson, 1992; Carr, 1998; Carr et al., 1992; Geske and Johnson, 1984; Huang et al., 1996; Ju, 1998; Ju and Zhong, 1999; MacMillan, 1986);
2. Trees (Breen, 1991; Cox et al., 1979; Komrad and Ritchken, 1991; Rendleman and Bartter, 1979);
3. Finite-differences (Brennan and Schwartz, 1977, 1978; Courtadon, 1982; Hull and White, 1990; Parkinson, 1977);
4. Finite-elements (Barone-Adesi et al., 2003; de Frutos, 2005, 2006);
5. Finite volumes(Zvan et al., 2001);
6. Stochastic dynamic programming (Ben-Ameur et al., 2006; Chen, 1970);

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7. Monte Carlo simulation (Boyle et al., 1997; Broadie and Glasserman, 1997).

The sandwich algorithms of Burkard et al. (1992) and Rote (1992) for convex functions based on secants and tangents cannot be applied directly in our context as they assume that the exact function values and their exact derivatives are known. However, we use the same concepts to envelop convex value functions of derivative contracts. Two approaches are used in the literature to derive upper and lower bounds for value functions of derivative contracts. In the first, closed-form envelopes are analytically derived (Broadie and Detemple, 1996; Davis et al., 2001; El Karoui et al., 1998; Johnson, 1983; Lévy, 1985). In the second, upper and lower bounds are computed by means of numerical procedures (Broadie and Cao, 2008; Broadie and Glasserman, 1997; Chung and Chang, 2007; Chung et al., 2010; Haugh and Kogan, 2004; Laprise et al., 2006; Magdon-Ismail, 2003).

The rest of the paper is organized as follows. In Section 2, we present our stochastic dynamic program and show how to obtain upper and lower bounds for convex value functions of derivative contracts. In Section 3, we provide a numerical investigation. Section 4 is a conclusion.

2. SDP formulation

2.1. Model and notation

Let $\{X\}$ be the price process of an underlying asset, interpreted here as the state process. Examples include stocks and interest rates. An American-style option is defined by a known payoff function $\varphi(t,x)\geq 0$ under exercise, where $t\in\{t_0=0<t_1<\ldots<t_N=T\}$ lies in a finite set of decision dates and $x=X_t$ is the level of the state variable at time t. We consider herein convex value functions on x. For example, $\varphi(t, x)=\max\,(0, K-x)$, for an American put option, where K is the option strike price. The payoff function $\varphi(t, x)$ is called the exercise value and indicated in the following by $v^e(t, x)$. A European option is a particular American-style option with a unique decision date at the option maturity date $t_N=T$, that is, $v^e(t_n, x)=0$, for $n<N$.

The option value function at t_n is defined by

$$v(t_n,x)=max\left(v^e(t_n,x),v^h(t_n,x)\right),\quad \text{for all } x. \tag{1}$$

where $v^h(t_n, x)$ is the holding value, which depends on the future potentialities of the contract. No-arbitrage pricing gives:

$$v^h(t_n,x)=E\left[\rho_n v\left(t_{n+1},X_{t_{n+1}}\right)\middle|X_{t_n}=x\right],\text{for all } x, \tag{2}$$

with the convention that $v^h(t_N, x)=0$, for all x. Here, and in the sequel, ρ_n is the (possibly stochastic) risk-free discount factor for the time period $\Delta t_n=t_{n+1}-t_n$ and $E\left[\cdot\middle|X_{t_n}=x\right]$ the conditional expectation symbol with respect to the risk-neutral probability measure. Eqs. (1) and (2) say that the option holder must decide, in an optimal way, whether or not to exercise its right at each decision date and level of the state variable. The option value function at maturity is

$$v(t_N,x)=v^e(t_N,x),\quad \text{for all } x. \tag{3}$$

American-style options cannot be evaluated in closed form, except for very few particular cases. Their values have to be approximated in some way. We propose herein upper and lower bounds, and obtain

$$\hat{v}(t_0,x_0)=\frac{\hat{v}_{sup}(t_0,x_0)+\hat{v}_{inf}(t_0,x_0)}{2}. \tag{4}$$

These bounds verify

$$\hat{v}_{\inf}(t,x)\leq v(t,x)\leq \hat{v}_{\sup}(t,x),\qquad \text{for all } t \text{ and } x,$$

which results in an approximation error $e\,(t, x)$ that verifies

$$0\leq e(t,x)\leq \hat{v}_{\sup}(t,x)-\hat{v}_{\inf}(t,x),\qquad \text{for all } t \text{ and } x. \tag{5}$$

Our approach provides not only an approximation for the option value (Eq. (4)) but also an upper bound for the approximation error (Eq. (5)).

Let $\mathcal{G}=\{a_1,\ldots, a_p\}$ be a mesh of grid points that covers the state space. We select the grid points to be the quantiles of the underlying asset at $t_N=T$. For example, in the Black and Scholes' model, the grid points are constructed as follows:

$$a_0=0,\ a_1=x_0e^{\left(r-\frac{\sigma^2}{2}\right)T-7\sigma\sqrt{T}},\ a_2=x_0e^{\left(r-\frac{\sigma^2}{2}\right)T-5\sigma\sqrt{T}},$$
$$a_{p-1}=x_0e^{\left(r-\frac{\sigma^2}{2}\right)T+5\sigma\sqrt{T}},\ a_p=x_0e^{\left(r-\frac{\sigma^2}{2}\right)T+7\sigma\sqrt{T}},\ a_{p+1}=\infty,$$

and

$$a_i=x_0e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}z_i}, \tag{6}$$

where z_i is the quantile of the standard normal distribution associated with fraction i/p, for $i=3,\ldots, p-2$. This construction, although not unique, has the property to set more evaluation nodes in the most visited areas (Eq. (6)). The grid points are kept fixed over time.

2.2. The upper envelope

Suppose that a convex upper envelope approximation $\tilde{v}^h_{sup}(t_{n+1},.)$ of $v^h(t_{n+1},.)$ is available at a certain future date t_{n+1}. One has

$$\tilde{v}^h_{\sup}(t_{n+1},x)\geq v^h(t_{n+1},x),\text{for all } x.$$

This assumption is not really a limitation since

$$\tilde{v}^h_{\sup}(t_N,x)=v^h(t_N,x)=0,\text{for all } x.$$

That is to say that SDP starts at $t_N=T$, where

$$\tilde{v}_{\sup}(t_N,x)=v(t_N,x)=v^e(t_N,x),\text{for all} x. \tag{7}$$

Eqs. (3) and (7) say that the SDP approximation $\tilde{v}_{\sup}(T_N,.)$ at maturity is nothing else than the true value function of the derivative to be evaluated. Now, we show how SDP moves backward in time from time t_{n+1} to time t_n.

First, we construct $\tilde{v}_{sup}(t_{n+1},x)$ as follows:

$$\tilde{v}_{sup}(t_{n+1},x)=max\left(v^e(t_{n+1},x),\tilde{v}^h_{sup}(t_{n+1},x)\right),\text{for all } x. \tag{8}$$

The value function in Eq. (8) is convex and over approximate $v(t_{n+1}, x)$, since the maximum of two convex functions is convex.

Then, we compute $\tilde{v}_{sup}(t_{n+1},a_k)$, for all a_k in $\mathcal{G}$, so that we can store the required information in a computer. Next, we use a piecewise linear interpolation, and extend $\tilde{v}_{sup}(t_{n+1},.)$ from $\mathcal{G}$ to the overall state space $\mathbb{R}_+$ as follows:

$$\hat{v}_{sup}(t_{n+1},x)=\sum_{i=0}^{p}\left(\alpha_i^{n+1}+\beta_i^{n+1}X_{t_{n+1}}\right)\mathbb{R}(a_i\leq x<a_{i+1}),\text{for all } x, \tag{9}$$

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