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# Fuzzy value-at-risk and expected shortfall for portfolios with heavy-tailed returns

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ABSTRACT

This paper is concerned with linear portfolio value-at-risk (VaR) and expected shortfall (ES) computation when the portfolio risk factors are leptokurtic, imprecise and/or vague. Following Yoshida (2009), the risk factors are modeled as fuzzy random variables in order to handle both their random variability and their vagueness. We discuss and extend the Yoshida model to some non-Gaussian distributions and provide associated ES. Secondly, assuming that the risk factors' degree of imprecision changes over time, original fuzzy portfolio VaR and ES models are introduced. For a given subjectivity level fixed by the investor, these models allow the computation of a pessimistic and an optimistic estimation of the value-at-risk and of the expected shortfall. Finally, some empirical examples carried out on three portfolios constituted by some chosen French stocks, show the effectiveness of the proposed methods.



## 1. Introduction

Value-at-risk (VaR) has become a standard tool for financial risk measure since its introduction by JP Morgan (1996). Its seminal form also known as Δ-normal VaR and defined as a percentile of the portfolio profit–loss distribution, is based on the assumption of normal distribution of the risk factors underlying the portfolio. The attractiveness of the Δ-normal VaR lies in its conceptual simplicity, ease of computation and implementation. However, each of the assumptions on which it relies is open to criticism, and it has been addressed numerous shortcomings by some risk management researchers.

The assumption relating to the normal distribution of the risk factors referred to financial assets returns have been extensively discussed in the literature over these last five decades. Some early studies such as Mandelbrot (1963) and Fama (1965) have showed that empirical returns exhibit higher peaks and heavy tails than would be predicted by a normal distribution, especially over short horizons. In the value-at-risk framework, other authors such as, Embrechts et al. (2002), Hosking et al. (2000), McNeil and Frey (1999) and Heyde (1999), using different financial data set, consistently highlighted systematic deviation from normality by finding high kurtosis and heavy tails. Motivated by this conclusion, several authors including Sadefo Kamdem and Genz (2008), Glasserman et al. (2002), Lopez and Walter (2000), Sadefo Kamdem (2005, 2009), introduced some portfolio value-at-risk models with heavy-tailed risk factors using different approaches. The Δ-normal VaR only measures percentiles of profit-loss distributions and does not provide any information about losses beyond the VaR level. Moreover, since the VaR is not subadditive, it is not a coherent risk measure. In order to remedy to this weakness, Artzner et al. (1999) introduced the expected shortfall (ES) defined as the conditional expected loss given that the loss is beyond the VaR level. The ES is a coherent risk measure [1] and allows taking into account the severity of an incurred damage event. This risk measure provides information about the thickness of the upper tail of the profit–loss distribution.

All the above-mentioned literature assumes that risk factors are real random variables crisply observed. Under the assumption of some well-known distribution of risk factors, the authors give some closed-form expressions of the VaR and ES or some numerical methods for their computation. The authors also assume that these risk factors are observed with precision and their probability distributions can be known based on these observations. However, in practice, the financial market is affected by imprecise observations, information insufficiency and expert's subjective opinions. The imprecision in observed risk factors is due to trading imperfections and microstructure noise.[2] Subjective opinions relate to linguistic imprecision induced by experts' judgments

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[1] See Acerbi and Tasche (2002) for a discussion on the coherence of the ES.
[2] Aït-Sahalia et al. (2011) noticed that these imperfections might be largely divided into three parts. The first represents the frictions inherent in the trading process: bid–ask bounces, discreteness of price changes and rounding, trades occurring on different markets or networks, etc. The second point concerns informational effects such as differences in trade sizes or informational content of price changes. The last point encompasses measurement or data recording errors.

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when associated to modeling. The information insufficiency is caused by the lack of knowledge regarding the current stocks market conditions and by the use of the sole return for summarizing the relative change of an asset's price over whole a period. Thus, the uncertainty of risk factors has two important sources: randomness and fuzziness. These two components of uncertainty can be jointly modeled by a fuzzy random variable (FRV) introduced by Kwakernaak (1978) and Puri and Ralescu (1986). An overview of the FRV is provided by Shapiro (2009). Following Mbairadjim Moussa et al. (2012), Smimou et al. (2008), Yoshida (2009) and some references therein, we use FRV to model risk factors in order to deal with both their randomness and imprecision. This paper aims at extending Yoshida (2009) who defined the VaR of a portfolio with imprecise risk factors using probabilistic expectation, evaluation weight and λ-function for estimating mean values, variances and the measurements of imprecision of fuzzy returns under the assumption of normal distribution. We introduce estimation models of VaR and ES portfolio under the assumption that the risk factors are fuzzy random variables in the sense of Puri and Ralescu (1986) and under heavy-tailed distributions.

The rest of the paper is organized as follows: In Section 2, we present the portfolio VaR and ES framework with an elliptic random vector of risk factors and we briefly review some basic concepts (fuzzy number, fuzzy random variable) of fuzzy set theory. Section 3 first discusses the Yoshida (2009) model's VaR portfolio before introducing some extensions with their corresponding ES and the use of heavy-tailed distributions. Section 4 is devoted to VaR and ES with heavy-tailed risk factors in the Puri and Ralescu (1986) framework. Sections 3 and 4 end with some numerical examples. Section 5 gives some concluding remarks. In addition, for a better understanding of the paper, a brief overview of elliptical distribution is given in Appendix A and the fuzzification process of risk factors is presented in Appendix B.

## 2. Portfolio model in uncertain environment

In this section, we recall some results of VaR and ES for a portfolio with elliptical risk factors and we briefly review some basic concepts of the fuzzy set theory.

### *2.1. Value-at-risk and expected shortfall*

We consider a linear portfolio of $n$ stocks or assets with value $\Pi(t)$ at time $t$. Its profit or loss over a time window $[0,t]$ is given by

$$\Delta\Pi(t) = \Pi(t) - \Pi(0) = \delta_1 X_t^1 + \delta_2 X_t^2 + \ldots + \delta_n X_t^n, \quad (1)$$

with $X_t^1, \ldots, X_t^n$ being the returns of its constituents over the same period and $\delta = (\delta_1, \delta_2, \ldots, \delta_n)$ the portfolio weights vector such that $\delta_1 + \delta_2 + \ldots + \delta_n = 1$ and $\delta_i \geq 0, \forall\, i = 1, \ldots, n$. As we only treat with the period $[0,t]$, the index $t$ will be omitted from formal expressions in the remainder of the paper.

The value-at-risk at a confidence level $1-\theta$ or risk level $\theta$, is given by the solution of the following equation

$$P\{\Delta\Pi(t) < -VaR_\theta\} = \theta. \quad (2)$$

Relation (2) follows the usual convention of recording portfolio losses by negative numbers, but stating the VaR as a positive quantity of money. The VaR so defined is related to the opposite of the rate of falling for portfolio assets.

We assume that $X = (X_1, \ldots, X_n)$ is elliptically distributed[3] with mean vector $\mu = (\mu_1, \ldots, \mu_n)$ and the covariance matrix $\Sigma = (\sigma_{ij})_{1 \leq i,j \leq n}$:

$$X \sim \mathcal{N}(\mu, \ \Sigma, \ g). \quad (3)$$

[3] A brief overview on multivariate elliptic distribution is available in Appendix A.

Here $\sigma_{ij} = cov(X_i, X_j)$ and $g$ is the density generator function.

Under assumption (3), Sadefo Kamdem (2005) has shown that the $VaR_\theta$ of $\Delta\Pi$ is given by

$$VaR_\theta = \sum_{i=1}^{n} \delta_i \mu_i + q_{\theta,n}^{g} \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \delta_i \delta_j \sigma_{ij}}, \quad (4)$$

where $q_\theta = q_{\theta,n}^{g}$ is the unique $g$-depending transcendental equation.[4]

For a given level $\theta$, the VaR does not provide information relating to the thickness of the distribution's upper tail and it is not coherent risk measure in the sense of Artzner et al. (1999). To bridge this gap, an alternative risk measure introduced by Artzner et al. (1997), is given by the expected shortfall (ES) (also known as Tail VaR or conditional VaR) as follows

$$ES_\theta = \mathbb{E}[-\Delta\Pi | -\Delta\Pi > VaR_\theta]. \quad (5)$$

Under the assumption of elliptical distribution of the above-mentioned risk factors, Sadefo Kamdem (2005) also proved the following results

$$ES_\theta = \sum_{i=1}^{n} \delta_i \mu_i + \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \delta_i \delta_j \sigma_{ij}} \times \frac{\pi^{\frac{n-1}{2}}}{\theta \Gamma\left(\frac{n+1}{2}\right)} \int_{(q_\theta)^2}^{\infty} \left(u - (q_\theta)^2\right)^{\frac{n-1}{2}} g(u) du, \quad (6)$$

where the special function $\Gamma$ is defined as follows:

$$\Gamma(u) = \int_0^{+\infty} t^{u-1} exp(-t) \ dt, \ u \in (0, +\infty). \quad (7)$$

### *2.2. Fuzzy numbers and fuzzy random variable*

Let $\mathbf{X}$ be a crisp set whose elements are denoted by $x$. A fuzzy subset $A$ of $\mathbf{X}$ is defined by its membership function $\mu_A : \mathbf{X} \to [0, 1]$ which associates each element $x$ of $\mathbf{X}$ with its membership degree $\mu_A(x)$ (Zadeh, 1965). The degree of membership of an element $x$ to a fuzzy set $A$ is equal to 0 (respectively 1) if one wants to express with certainty that $x$ does not belong (respectively belongs) to $A$.

The crisp set of elements that belong to the fuzzy set $A$ at least to the degree $\alpha$ is called the $\alpha$-cut or $\alpha$-level set and defined by:

$$A_\alpha = \{x \in X | \mu_A(x) \geq \alpha\}. \quad (8)$$

$A_0$ is the closure[5] of the support[6] of $A$.

Fuzzy numbers have fuzzy properties, examples of which are the notions of "around ten percent" and "extremely low". Dubois and Prade (1980, p. 26) characterize the fuzzy numbers as follows.

**Definition 2.1.** A fuzzy subset A of $\mathbb{R}$ with membership $\mu_A : \mathbb{R} \to [0, 1]$ is called fuzzy number if

1. A is normal, i.e. $\exists\, x_0 \in \mathbb{R} | \mu_A(x_0) = 1$;
2. A is fuzzy convex, i.e. $\forall\, x_1, x_2 \in \mathbb{R} \,|\, \mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_A(x_1), \mu_A(x_2)\}, \forall\, \lambda \in [0, 1]$;
3. $\mu_A$ is upper semi continuous[7];
4. supp(A) is bounded.

[4] For a detailed discussion on the $g$-depending transcendental equation, see Sadefo Kamdem (2009).

[5] The closure of the support of $A$ is the smallest closed interval containing the support of $A$ (Shapiro, 2009).

[6] The support of $A$ is the set of all $x$ such that $\mu_A(x) > 0$. (Shapiro, 2009)

[7] Semi-continuity is a weak form of continuity. Intuitively, a function $f$ is said to be upper semi-continuous at point $x_0$ if the function values for arguments near $x_0$ are either close to $f(x_0)$ or less than $f(x_0)$.