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Parameter identification for fractional Ornstein–Uhlenbeck processes based on discrete observation

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ABSTRACT

Fractional Ornstein–Uhlenbeck process is an extended model of the traditional Ornstein–Uhlenbeck process that provides some useful models for many physical and financial phenomena demonstrating long-range dependencies. Obviously, if some phenomenon can be modeled by fractional Ornstein–Uhlenbeck processes, the problem of estimating unknown parameters in these models is of great interest, especially, in discrete time. This paper deals with the problem of estimating the unknown parameters in fractional Ornstein–Uhlenbeck processes. The estimation procedure is built upon the marriage of the quadratic variation method and the maximum likelihood approach. The consistency of these estimators is also provided. Simulation outcomes illustrate that our methodology is efficient and reliable.

1. Introduction

The Ornstein–Uhlenbeck process, which is also called the Vasicek model, is being extensively used in finance these days as a one-factor short-term interest rate model. Moreover, this process has found many applications in fields as diverse as economics and finance, biology, physics, chemistry, medicine and environmental studies. However, all models involve unknown parameters or functions, which need to be estimated from observations of the process. The estimation of these processes is therefore a crucial step in all applications, in particular in applied finance. In the case of Ornstein–Uhlenbeck processes driven by Wiener processes, the statistical inference for these processes has been studied earlier and a comprehensive survey of various methods was given in Prakasa Rao (1999) and Bishwal (2008). Thanks to the Markovian and Gaussian properties of Wiener processes, both the maximum likelihood estimator (MLE) and the least squares estimator (LSE) are easy to obtain and exhibit asymptotic unbiasedness, efficiency and normality under the usual regularity conditions (see Bishwal, 1999, 2010). For MLE and LSE based on discrete observations, we refer to Bishwal (2001), Bishwal and Bose (2001), Aït-Sahalia (2002) and the references therein.

Recently, long memory processes have been used for modeling various stochastic phenomena that arises in areas such as hydrology, geophysics, medicine, genetics, internet traffic patterns and financial economics. The most popular stochastic process that exhibits long-range dependence is of course the fractional Brownian motion (fBm). Consequently, statistical inference problems related to processes driven by the fBm have been studied extensively by many authors (see, for instance Bishwal, 2003; Bishwal, 2008; Hu and Nualart, 2010; Kleptsyna and Le Breton, 2002; Lee and Song, 2013; Tudor and Viens, 2007). An extensive review on most of the recent developments related to the parametric and other inference procedures for stochastic models driven by fBm can be found in Prakasa Rao (2010).

These papers above focused on the problem of estimating the unknown parameters of stochastic models driven by fBm in the continuous-time case. However, as has been stressed by various authors, the continuous sampling observations hypothesis is unreasonable since in practice it is obviously impossible to observe a process continuously over any given interval, due, for instance, to the limitations on the precision of the measuring instrument or the unavailability of observations at every point in time (Prakasa Rao, 1999). For example, in practice it is usually only possible to observe these processes in discrete-time samples (e.g., stock prices collected once a day). Therefore, statistical inference for discretely observed diffusions was of great interest for practical purposes and at the same time it posed a challenging problem. Thus, the most recent research in stochastic processes estimation has been concerned with discrete time observations, where some progress has been made, both in parametric and nonparametric estimation. Actually, in recent years, there were some attempts to investigate the inference problems of stochastic processes associated with fBm, when the observational scheme is discrete in nature. For instance, Tudor and Viens (2007), Bishwal (2011), Xiao et al. (2012) and Hu et al. (2012) treated estimation problem of the drift parameter in stochastic models associated with fBm using the following idea:

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first, one constructs an estimator in the continuous time model and then the continuous estimator is discretized. Some author (see, for example, Bertin et al., 2011; Hu et al., 2011, Rifo et al., 2013, Sun et al., 2013; Xiao et al., 2011; Xu et al., 2012) considered the problem of estimating coefficients in some models driven by fBm based on an approximation by a discretized stochastic differential equation. More recently, Kaur et al. (2011) investigated the parameter estimation problem in stochastic differential equation driven by fBm using an approximation of the continuous-time log-likelihood function. Some authors have studied the parameter estimation problem of the stochastic models driven by fBm using the method of Malliavin calculus (Chronopoulou and Tindel, 2013) and the approach of generalized moment-matching (Papavasiliou and Ladroue, 2011).

Since fBm is not a Markov process, the Kalman filter method cannot be applied to estimate the parameters of stochastic processes driven by fBm. Consequently, it is a convenient way to handle the estimation problem by replacing fBm with its associated disturbed random walk. This method, using a random walk approximation of the fBm, was originally introduced by Sottinen (2001). Moreover, this Donsker type approximation approach was further extended by Bertin et al. (2011), who estimated parameters in stochastic processes of the Gaussian case (fBm with drift) and the non-Gaussian case (the Rosenblatt process with drift). In this paper, we shall consider the parameter estimation problem for a special fractional process, namely fractional Ornstein–Uhlenbeck process (FOUP). The main contribution of this paper is to determine the estimators for the FOUP and to show the strong consistence of these estimators. For these purposes, we first construct the estimators for FOUP, which is observed at discrete points of time. Then we present the asymptotic behavior of these estimators. Finally, we describe the numerical implementation based on our method and offer some numerical results to test the accuracy and validity of our proposal.

Our paper is organized as follows. In Section 2 we propose estimators for FOUP from discrete observations. The almost sure convergence of these estimators is also provided in the latter part of this section. Section 3 provides Monte-Carlo experiments to test the performance of the estimators under different sampling conditions. Finally, Section 4 includes conclusions and directions of further work.

2. Estimation procedure

The Ornstein–Uhlenbeck process was proposed by Uhlenbeck and Ornstein in a physical modeling context, as an alternative process to Brownian motion. Since the original paper appeared, the model has been used in a wide variety of application areas. In finance, it is best known in connection with the Vasicek interest rate model. However, it is well-known that many time series, in diverse fields of application, may exhibit the phenomenon of long-memory. The most popular stochastic process with long-memory property is of course the fBm. Actually, a crucial problem with the applications of stochastic processes driven by fBm in practice is how to obtain the unknown parameters in these stochastic models. Consequently, the topic of parameter estimation for stochastic differential equations driven by fBm has been widely studied. In the following, we will deal with the problem of estimating the unknown parameters in the FOUP.

2.1. Model specification

Now, let $(\Omega,\mathcal{F},\mathcal{F}_{t\geq 0},\mathbb{P})$ be a stochastic basis satisfying the usual conditions. The natural filtration of a process is understood as the $\mathbb{P}$-completion of the filtration generated by this process. In this paper, we consider the estimation problem of the following stochastic differential equation from discrete observations

$$dX_t=-\theta X_t dt+\sigma dB_t^H\ ,t\geq 0, \tag{1}$$

where the drift parameter θ is strictly positive, σ is a constant and B_t^H is a fBm with Hurst parameter $H>1/2$ on some probability space $(\Omega,\mathcal{F},\mathbb{P})$ with a filtration $\{\mathcal{F}\}_{t\geq 0}$.

It is worth emphasizing that the solution of Eq. (1) is given by

$$X_t=X_0-\theta\int_0^t X_s ds+\sigma B_t^H\ ,t\geq 0\ .$$

Here the unknown parameter $\vartheta=(\theta,\sigma,H)$ belongs to an open subset Θ of $(0,+\infty)\times(0,+\infty)\times(\frac{1}{2},1)$. The FOUP is useful since it presents the long-range dependence and it produces a burstiness in the sample path behavior. The present work exposes an estimation procedure for estimating all three components of ϑ given the regular discretization of the sample path. Now, we use the following step. Let $\{X_t;t\quad\mathbb{R}\}$ be the FOUP with $H>\frac{1}{2}$ and suppose that the data X_t is recorded discretely at points $(h,2h,\cdots,Nh)$ in the time interval $[0,T]$, where $h=\frac{T}{N}$ is the step size of the partition and T is the time span of the data. For simplicity, we assume that $T=1$. Thus, the full sequence of N observations is $\{X_h,X_{2h},\cdots,X_{Nh}\}=\left\{X_{\frac{1}{N}},X_{\frac{2}{N}},\cdots,X_{\frac{N}{N}}\right\}$.

2.2. The estimation procedure

Based on the above situation of discrete observations, we now proceed to estimate unknown parameters of H, σ and θ based on quadratic variation method and maximum likelihood approach. First, using the result of Kubilius and Melichov (2010), we can state that the estimator of Hurst parameter in FOUP can be written as

$$\hat{H}=\frac{1}{2}-\frac{1}{2\ln 2}\ln\frac{\sum_{i=1}^{N-1}\left[X_{(i+1)h}-X_{ih}\right]^2}{\sum_{i=1}^{\lfloor\frac{N}{2}\rfloor-1}\left[X_{2(i+1)h}-X_{2ih}\right]^2}\ , \tag{2}$$

where $\lfloor z\rfloor$ denotes the greatest integer not exceeding z.

Then, from Eq. (1), it is clear that the diffusion parameter σ^2 can be (at least theoretically) computed on any finite time interval. Thus, we can obtain an estimator for the diffusion parameter by using the quadratic variation

$$\hat{\sigma}^2=\frac{\sum_{i=1}^{N-1}\left(X_{(i+1)h}-X_{ih}\right)^2}{(N-1)h^{2H}}\ . \tag{3}$$

Hence we can estimate H and σ^2 from any small interval as long as we have a enough observation of the process. Finally, we are in a position to estimate the drift parameter. We would like to mention that, since the increments of the process B_t^H are not independent anymore and the process B_t^H is not a semimartingale, the martingale type techniques cannot be used to study this estimator. This problem will be avoided by the use of the random walks that approximate B_t^H.

We will make use of the following representation (see, for instance, Sottinen, 2001).

Lemma 2.1. *The fBm with Hurst parameter $H>1/2$ can be represented by its associated disturbed random walk*

$$B_t^{H,N}=\sum_{i=1}^{Nt}\sqrt{N}\left(\int_{\frac{i-1}{N}}^{\frac{i}{N}}K^H\left(\frac{Nt}{N},s\right)ds\right)\varepsilon_i \tag{4}$$

with

$$K^H(t,s)=c_H\left(H-\frac{1}{2}\right)s^{\frac{1}{2}-H}\int_s^t(u-s)^{H-\frac{3}{2}}u^{H-\frac{1}{2}}du \tag{5}$$

where c_H is the normalizing constant

$$c_H=\sqrt{\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)}}\ , \tag{6}$$