Economic Modelling 36 (2014) 608–611

Contents lists available at ScienceDirect

Economic Modelling

journal homepage: www.elsevier.com/locate/ecmod

Optimal growth theory: Challenging problems and suggested answers ☆

O. de La Grandville

Stanford University, Department of Management Science and Engineering, 475 Via Ortega, Stanford, CA 94305, United States

ARTICLE INFO

Keywords:
Optimal growth theory
Competitive equilibrium

ABSTRACT

We argue that optimal economic growth is confronting serious applicability problems, having nothing to offer in these days of high public deficits accompanied by high unemployment rates. In particular, the theory is not capable of indicating optimal savings rates; those are systematically in ranges that can be considered as unacceptable, or are accompanied by unrealistically high real growth rates. Faulty is the systematic use of arbitrary utility functions, which turn out to be contradictory to competitive equilibrium. We then show how relying on the hypothesis of competitive equilibrium yields reasonable, perfectly acceptable numbers for the optimal savings rate.

1. Introduction

In his *Introduction to History* (1377), Ibn Khaldun (1958) pronounced excess of consumption in any society as leading to a decline in civilization. In these years of economic turmoil marked in many countries by a decrease in the overall savings rate, high deficits as well as high unemployment rates, one may well wonder why the question of an optimal savings or investment rate has not yet come to the foreground. This paper will argue that the fundamental question of optimal growth theory, i.e. the determination of optimal time paths for the economy, namely consumption and investment, has in fact never been properly answered. We will show that the reasons for this dire situation pertain to the very way the problem has been posed until now. We will then suggest a possible solution.

2. The traditional methodological setting and its associated challenges

The theoretical problem of economic growth, as it has been considered until now, is apparently as straightforward to formulate as it is easy to solve. To demonstrate the latter assertion we will first solve the problem using direct economic reasoning; we will then give a formal proof.

Suppose we want to find the optimal trajectory of capital, denoted $\vec{K}^*$, such that society maximizes the sum of discounted consumption flows

$$W = \int_0^\infty U(C_t) e^{-\int_0^t i(z)dz} dt, \tag{1}$$

where $U(C)$ is strictly concave and where the problem is constrained by

$$C_t = Y_t - \dot{K} = F(K_t G_K(t), L_t G_L(t)) - \dot{K}; \tag{2}$$

Y_t stands for net income (net of capital depreciation); $G_K(t)$ and $G_L(t)$ are capital – and labor – augmenting technical progress functions; $F(K_t G_K(t), L_t G_L(t))$ is strictly concave with respect to its arguments (U,V) defined by $K_t G_K(t) \equiv U$, $L_t G_L(t) \equiv V$.

2.1. An immediate, intuitive derivation of the optimal rule of capital accumulation

It is easy to figure out that investing today in capital goods must be exactly compensated by rewards in the future. Therefore it should be possible to determine an optimal trajectory of capital (or an optimal time path of investment, its derivative) through the following reasoning. Suppose that to any flow of consumption C_t is associated a utility flow $U(C)$ with the above-mentioned properties. Then an optimal trajectory must be such that, at any time t, the cost – or sacrifice – of investing one unit, equal to the marginal utility of consumption, must be equal to the sum from time t to infinity of all possibly future rewards carried by this investment. It should then be such that the following equality is maintained at each point of time t:

Sacrifice at time t = Sum of future rewards from time t to infinity

which translates as

$$U'(C_t) = \int_t^\infty U'(C_\tau) \frac{\partial F}{\partial K}(K_\tau G_K(\tau), L_\tau G_L(\tau)) e^{-\int_t^\tau i(z)dz} d\tau. \tag{3}$$

☆ This essay is an outgrowth of an invited lecture at the University of Rome in May 2012. I am deeply grateful to Kenneth Arrow, Michael Binder, Giuseppe De Archangelis, Bjarne Jensen, Rainer Klump, Ulrich Kohli, Peter McAdam, Bernardo Maggi, Enrico Saltari, Robert Solow, Wolfgang Stummer, Juerg Weber, Alpo Willman and Milad Zarin-Nejadan, as well as to the participants in the conference in Rome and my students at Stanford for very stimulating remarks.

E-mail address: ola@stanford.edu.

0264-9993/$ – see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.econmod.2013.02.019

where $\exp(-\int_t^\tau i(z)dz)$ is the discounting factor from t to τ of the utility flow received by society at any time τ between t and infinity[1].

Differentiating Eq. (3) with respect to t gives

$$i(t) = \frac{\partial F}{\partial K}(K_t G_K(t), L_t G_L(t)) + \frac{\dot{U}'(C_t)}{U'(C_t)}, \tag{4}$$

the differential equation governing the motion of capital. That the problem is correctly solved through this direct reasoning will now be easily confirmed.

2.2. A formal derivation

Applying the Euler equation to (1) where $C(t)$ is replaced by $F(K_t G_K(t), L_t G_L(t)) - \dot{K}$, we just solve the unconstrained problem

$$\text{Max}_{\overline{K}} W = \int_0^\infty U\left[F(K_t G_K(t), L_t G_L(t)) - \dot{K}\right] e^{-\int_0^t i(z)dz} dt \tag{5}$$

Denoting $N\left(K, \dot{K}, t\right)$ the integrand of the above integral and applying the Euler equation $\partial N/\partial K - (d/dt)\left(\partial N/\partial \dot{K}\right) = 0$ yields Eq. (4), a necessary and sufficient condition for maximizing W (by Takayama's theorem since both $U(.)$ and $F(.)$ are concave).

2.3. The challenges

We should first observe that the inocuous looking Eq. (4) is a second-order, nonlinear differential equation in K. Using the fact that $C_t = F(K_t G_K(t), L_t G_L(t)) - \dot{K}$, we have

$$\dot{U}'(C_t) = U''\left[F(K_t G_K(t), L_t G_L(t)) - \dot{K}\right] . \left\{F'_K \dot{K} + F'_{G_K} \dot{G}_K + F'_L \dot{L} + F'_{G_L} \dot{G}_L - \ddot{K}\right\},$$

and therefore Eq. (4) is to be written as

$$i(t) = \frac{\partial F}{\partial K}(K_t G_K(t), L_t G_L(t)) + \frac{U''\left[F(K_t G_K(t), L_t G_L(t)) - \dot{K}\right]}{U'\left[F(K_t G_K(t), L_t G_L(t)) - \dot{K}\right]} \times \left\{F'_K \dot{K} + F'_{G_K} \dot{G}_K + F'_L \dot{L} + F'_{G_L} \dot{G}_L - \ddot{K}\right\}. \tag{6}$$

Here is where a first, technical, difficulty lies. There is no chance that Eq. (6) yields an analytic solution; the same applies to the corresponding first-order, nonlinear system of equations in (K,C) that can be derived from Eqs. (2) and (4):

$$\dot{K} = F(K_t G_K(t), L_t G_L(t)) - C_t \tag{7}$$

$$\dot{C} = \frac{U'(C)}{U''(C)} . \left[i(t) - F'_K(K_t G_K(t), L_t G_L(t))\right] \tag{8}$$

Only numerical analysis will provide a clue to the two following concomitant problems:

a) Determining whether, for any kind of utility or production function, the economy will converge asymptotically toward an equilibrium, or whether the economy is doomed, heading toward disaster either by consuming its existing capital or by investing so much that consumption collapses to zero, and
b) determining the initial value C_0 leading to equilibrium.

[1] Why shouldn't we have at time t, rather than Eq. (3), the *inequality* $\int_t^\infty U'(C_\tau) \frac{\partial F}{\partial K}(K_\tau G_K(\tau), L_\tau G_L(\tau)) e^{-\int_t^\tau i(z)dz} d\tau > U'(C_t)$? If that inequality applied, it would mean that we would not have invested enough; indeed, we would have foregone reaping a potential benefit. Investing one unit more rather than consuming it would decrease consumption and increase the right-hand side of Eq. (4). Symmetrically, investing more would decrease the integrand in the left hand. Such a process would have to operate until both sides are equal.

It turns out that in the days of the revival in optimal growth theory (the sixties), solving such equations numerically was quite laborious, requiring the use of main-frame computers. Carrying out a complete qualitative analysis of the stability of Eq. (6) or its corresponding system of nonlinear, first-order Eqs. (7), (8), would have – and probably did – put off even the strongest willed. Today however, many softwares enable to solve such systems in a few seconds; there is then no excuse not to examine closely what these equations imply in terms of possible equilibria and implied savings rates under different assumptions.

A wide array of production functions may be used; one of the most popular is the CES. We note here that for many years it has been supposed – unfortunately for no good reason – that technical progress was solely labor-augmenting. In fact, capital-augmenting technical progress should definitely be part of the basic premises. As to the utility function, three families have been and still are considered as appropriate: the power function C^α, $\alpha < 1$, the log function $\log C$, and the negative exponential $(-1/\beta)C^{-\beta C}$. The two first functions can be collapsed into the single formulation $(C^\alpha - 1)/\alpha$, because $\lim_{\alpha \to 0} (C^\alpha - 1)/\alpha = \log C$, and also because giving an affine change to the utility function (in this case multiplying C^α by $1/\alpha$ and adding the constant $-1/\alpha$) does not change the Euler equation resulting from the optimization problem.

There is of course no justification why society as a whole should be considered as valuing consumer goods or services with these functions, and it seems that the only reason they were declared fit for service stems from their simplicity, leading to simple expressions for the term $U'(C)/U''(C)$ in Eq. (8), equal to $C/(\alpha - 1)$ in the power case and to $-\beta$ in the negative exponential case. That type of arbitrariness could be itself a matter of concern. But additional predicaments are to be encountered.

First, it has repeatedly been been shown, through numerical analysis, that it is a hopeless venture to obtain reasonable growth or saving paths by having recourse to such utility functions. This goes all the way back to Frank Ramsey (1928) who, in his path-breaking essay "A Mathematical Theory of Saving" (1928), trying to put some numbers on his own model, recognized that "the rate of saving which the rule requires is greatly in excess of that which anyone would suggest […] the amount that should be saved out of a family income of £500 would be about £300". He added that the concave utility function he used was "put forward merely as an illustration" (*op. cit.*, p. 548) but one can imagine that he was disappointed by the result. Did he, at the time, try to do what many would have done in such circumstances, i.e. change the utility function? We will never know.

Three decades later, at the time of the renewal of interest in the topic, Richard Goodwin (1961) in his essay "The Optimal Growth Path for an Underdeveloped Economy"(1961) considered two different growth models; in each of those, at some point of time the optimal savings rate reached unacceptable levels, also in the order of 60%. (We note here that in a 2006 conference, Robert Solow declared he remembered very well reading the Goodwin paper just before or just after its publication, and being very worried about such high levels of the optimal savings rate).

For our part, with the generous help of our colleague Ernst Hairer we have carried out (2009, Chapter 10, pp. 236–257) a thorough qualitative test of these utility functions, using the standard model with labor-augmenting technical progress. For the economy to be on the stable arm leading to equilibrium, the power functions implied huge, unacceptable initial savings rate whenever α was in the interval [0,1).

To obtain reasonable initial savings values, one has to resort to negative values for α; but in turn this completely changes the nature of the utility function: first, it becomes very quickly bounded by the asymptote given by $\lim_{C \to \infty}(C^\alpha - 1)/\alpha = -1/\alpha$, $\alpha < 0$; secondly, the marginal utility decreases at an absurdly high speed. Consider for instance $\alpha = -3$ (by all means not an exceedingly negative value; we have seen much lower values in the literature). The asympote of the utility function is 1/3; this value is reached extremely quickly: already for $C = 3$, $U(C) = 0.32$, and for the following values of C: 0.5, 1.5 and 1.5, the marginal utility $U'(C)$ is lowered from 1.12,