Economic Modelling 33 (2013) 159–163

Contents lists available at SciVerse ScienceDirect

Economic Modelling

journal homepage: www.elsevier.com/locate/ecmod

Some results about the exponential ordering of inactivity time

M. Kayid [*,1], L. Alamoudi

King Saud University, College of Science, Dept. of Statistics and Operations Research, P.O. Box 2455, Riyadh 11451, Saudi Arabia

ARTICLE INFO

Article history:
Accepted 3 April 2013

Keywords:
Exponential order
Convolution
Mixture
Shock models

ABSTRACT

The purpose of this paper is to study new notions of stochastic comparisons and aging classes based on the exponential order. We provide some preservation properties of the exponential order of inactivity time under the reliability operations of convolution and mixture. Some applications to shock models are discussed.

1. Introduction

It is well established that stochastic comparison relations constitute an important tool in the analysis of various reliability and actuarial problems. In the context of reliability theory these comparisons have been found useful for modeling, or the design of better systems (cf. Goovaerts et al., 1990, Kaas et al., 1994 and Shaked and Shanthikumar, 2007). Most stochastic comparison relations used in reliability and actuarial sciences are of integral form. An integral stochastic comparison $\leq_{\mathcal{F}}$ is a stochastic order relation generated by a class $\mathcal{F}$ of measurable functions. To be specific, given two random variables X and Y, X is said to precede Y in the $\mathcal{F}$-sense, written as $X \leq_{\mathcal{F}} Y$ iff

$$E[\varphi(X)] \leq E[\varphi(Y)] \quad \text{for all the functions } \varphi \in \mathcal{F}, \tag{1.1}$$

provided the expectations involved in Eq. (1.1) exist. Such stochastic comparisons have been studied by Marshall (1991) and Müller (1997) in a very general setting. Note that $X \leq_{\mathcal{F}} Y$ in fact means that the distribution functions F_X and F_Y corresponding to X and Y are ordered, and not the particular versions of these random variables. One of these comparisons is the exponential order, that we recall here.

Let X and Y be two random variables with distributions F_X and F_Y, and denote their survival functions by $\overline{F}_X = 1 - F_X$ and $\overline{F}_Y = 1 - F_Y$, respectively. Their exponential functions are defined as, for all $s > 0$

$$\psi_X(s) = E[exp(sX)] \quad \text{and} \quad \psi_Y(s) = E[exp(sY)] \,.$$

Given two random variables X and Y, X is said to be smaller than Y in the exponential order (denoted as $X \leq_{exp} Y$) if $Ee^{t_0 Y}$ is finite for some $t_0 > 0$, and

$$\psi_X(s) \leq \psi_Y(s) \quad \text{for all } s > 0.$$

Notice also that

$$X \leq_{exp} Y \Leftrightarrow \int_0^\infty e^{su}\overline{F}_X(u)du \leq \int_0^\infty e^{su}\overline{F}_Y(u)du \quad \text{for all } s > 0,$$

which possesses some applications in reliability theory. From a probabilistic point of view, the exponential order boils down to requiring that the moment generating functions of the non-negative random variables X and Y are uniformly ordered. The exponential order also expresses the common preferences among risks of all the decision-makers with a utility function of the form $\varnothing(x) = 1 - \exp(-sx)$, $s > 0$. In an actuarial framework, exponential order possesses several interpretations. For example, consider an insurance company setting its price list according to the exponential premium calculation principle. In such a case, the amount of premium $\pi_s[X]$ relating to the risk X is given by

$$\pi_s[X] = \frac{1}{s} lnE[exp(sX)]. \tag{1.2}$$

From Eqs. (1.1) and (1.2), we obviously have that

$$X \leq_{exp} Y \Leftrightarrow \pi_s[X] \leq \pi_s[Y] \quad \text{for all } s > 0.$$

Other interpretations, properties and applications of the exponential order can be found in Klar and Müller (2003) and Denuit (2001). In many reliability engineering problems, it is of interest to consider

* Corresponding author.
E-mail address: drkayid@ksu.edu.sa (M. Kayid).
1 Permanent address: Department of Mathematics, Faculty of Science, Suez University, Egypt.

0264-9993/$ – see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.econmod.2013.04.002

variables of the kind $X_{(t)} = [t - X|X \leq t]$, for fixed $t \in (0,l_X)$ and $l_X = sup\{t : F_X(t) < 1\}$, having distribution function $F_{(t)}(s) = P[t - X \leq s| X \leq t]$, and known in literature as inactivity time (Kayid and Ahmad, 2004, Ahmad and Kayid, 2005, Ahmad et al., 2005, Izadkhah and Kayid, 2013 and Kayid et al., 2013). To compare the exponential order of inactivity time, another ordering has come to use in reliability and life testing problems (see Al-amoudi, 2011). Its definition is as follows:

Definition 1.1. A nonnegative random variable X is said to be smaller than Y in the exponential order of inactivity time order (denoted by $X \leq_{exp\text{-}it} Y$) iff

$$X_{(t)} \geq_{exp} Y_{(t)} \quad \text{for all } t \in (0, l_X) \cap (0, l_Y). \tag{1.3}$$

On the other hand, some authors have made efforts to investigate the characterizations of some non-parametric classes by means of various stochastic orders. For example, let us denote $X_t = [X - t|X > t]$, with $t \in (0,l_X)$ the residual life of the random variable X, i.e., the variable having $F_t(s) = P[X - t \leq s|X > t]$ as its distribution function (Barlow and Proschan, 1981). Similarly denote $X_s = [X - s|X > s]$, and l_Y. If $\digamma$ denotes some aging property, a general procedure to give definitions and characterizations of $\digamma$ (as has been pointed out in Pellerey and Shaked, 1997) is by means of stochastic orders of residual lifetimes of the form

$$X \in \digamma \Leftrightarrow X_s \leq_{st-ord} X_t \quad \text{for all } s \leq t, \ s, t \in (0, l_X)$$

or

$$X \in \digamma \Leftrightarrow X_t \leq_{st-ord} X \ \text{ for all } t \in (0, l_X),$$

Where $\geq_{st\text{-}ord}$ denotes some stochastic orders.

Recently, Al-amoudi (2011) proposed new aging class following the previous procedures for the exponential order.

Definition 1.2. A nonnegative random variable X is said to have decreasing residual live in the exponential order (denoted as $X \in$ DRLE) if

$$X_t \geq_{exp} X_s, \quad 0 \leq t < s.$$

In the current investigation, we enhance the study of the exponential order of inactivity time (*exp-it*). In Section 2, we present some preservation results of the *exp-it* order under the operations of convolution and mixtures. In Section 3, we provide some applications to shock models and describe some simple examples of applications in recognizing situations where the random variables are comparable according to this order. Finally, in Section 4, we provide a brief conclusion, and some remarks about current and future research.

Throughout this paper, the term increasing is used instead of monotone non-decreasing and the term decreasing is used instead of monotone non-increasing. We also assume that all random variables under consideration are absolutely continuous and have 0 as the common left end-point of their supports, and all expectations are implicitly assumed to be finite whenever they appear.

2. Preservation properties

Preservation properties of an order under some reliability operations are of importance in reliability theory (Barlow and Proschan, 1981). This section deals with preservation properties of the *exp-it* order under some reliability operations such as mixture and convolution. Given two random variables X and Y, let us denote for all $s > 0$

$$\psi^*_{X_{(t)}}(s) = \frac{\int_0^t e^{-su} F_X(u) du}{e^{-st} F_X(t)} \quad \text{and} \quad \psi^*_{X_{(t)}}(s) = \frac{\int_0^t e^{-su} F_Y(u) du}{e^{-st} F_Y(t)}.$$

Observe that, by the Definition 1.1, it holds

$$X \leq_{exp-it} Y \Leftrightarrow \psi^*_{X_t}(s) \geq \psi^*_{X_t}(s), \quad \text{for all } t; s > 0.$$

An equivalent condition for *exp-it* order is given in the following result.

Proposition 2.1. *Let X and Y two continuous non-negative random variables. Then*

$$X \leq_{exp-it} Y \Leftrightarrow \frac{\int_0^t e^{-su} \ F_X(u) du}{\int_0^t e^{-su} \ F_Y(u) du} \text{ is decreasing in } t \in (0, l_X) \cap (0, l_Y), \text{ for all } s > 0.$$

Proof. Let us observe that

$$\psi^*_{X_{(t)}}(s) = \frac{\int_0^t e^{-su} F_X(u) du}{e^{-st} F_X(t)} = \frac{\int_0^t e^{-su} F_X(u) du}{(\partial/\partial t)\left(\int_0^t e^{-su} F_X(u) du\right)}; \tag{2.1}$$

therefore given $s > 0$, by Eqs. (1.1) and (2.1)

$$X \leq_{exp-it} Y \Leftrightarrow \frac{\int_0^t e^{-su} F_X(u) du}{(\partial/\partial t)\left(\int_0^t e^{-su} F_X(u) du\right)} \geq \frac{\int_0^t e^{-su} F_Y(u) du}{(\partial/\partial t)\left(\int_0^t e^{-su} F_Y(u) du\right)}$$

$$\Leftrightarrow \frac{\int_0^t e^{-su} F_X(u) du}{\int_0^t e^{-su} F_Y(u) du} \geq \frac{(\partial/\partial t)\left(\int_0^t e^{-su} F_X(u) du\right)}{(\partial/\partial t)\left(\int_0^t e^{-su} F_Y(u) du\right)}$$

$$\Leftrightarrow \frac{\int_0^t e^{-su} F_X(u) du}{\int_0^t e^{-su} F_Y(u) du} \text{ is decreasing in } t \in (0, l_X) \cap (0, l_Y). ||$$

□

The next result presents a preservation property of the *exp-it* order under convolution.

Theorem 2.1. *Let X_1, X_2 and Y be three non-negative random variables, where Y is independent of both X_1 and X_2, and let Y have density g. If $X_1 \leq_{exp\text{-}it} X_2$ and g is log-concave then $X_1 + Y \leq_{exp\text{-}it} X_2 + Y$.*

Proof. Because of Proposition 2.1, it is enough to show that for all $0 \leq t_1 \leq t_2$ and $x > 0$

$$\frac{\int_0^\infty \int_{-\infty}^{t_1} e^{sx} P[X_1 \leq u - x] g(t_1 - u) du \ dx}{\int_0^\infty \int_{-\infty}^{t_1} e^{sx} P[X_2 \leq u - x] g(t_1 - u) du \ dx} \geq \frac{\int_0^\infty \int_{-\infty}^{t_2} e^{sx} P[X_1 \leq u - x] g(t_2 - u) du \ dx}{\int_0^\infty \int_{-\infty}^{t_2} e^{sx} P[X_2 \leq u - x] g(t_2 - u) du \ dx}.$$

Since Y is non-negative then $g(t - u) = 0$ when $t < u$, hence the above inequality is equivalent to

$$\frac{\int_0^\infty \int_{-\infty}^\infty e^{sx} F_{X_1}(u - x) g(t_1 - u) dudx}{\int_0^\infty \int_{-\infty}^\infty e^{sx} F_{X_2}(u - x) g(t_1 - u) dudx} \geq \frac{\int_0^\infty \int_{-\infty}^\infty e^{sx} F_{X_1}(u - x) g(t_2 - u) dudx}{\int_0^\infty \int_{-\infty}^\infty e^{sx} F_{X_2}(u - x) g(t_2 - u) dudx}$$

for all $0 \leq t_1 \leq t_2$, or equivalently,

$$\begin{vmatrix} \int_0^\infty \int_{-\infty}^\infty e^{sx} F_{X_2}(u - x) g(t_2 - u) dudx & \int_0^\infty \int_{-\infty}^\infty e^{sx} F_{X_1}(u - x) g(t_2 - u) dudx \\ \int_0^\infty \int_{-\infty}^\infty e^{sx} F_{X_2}(u - x) g(t_1 - u) dudx & \int_0^\infty \int_{-\infty}^\infty e^{sx} F_{X_1}(u - x) g(t_1 - u) dudx \end{vmatrix} \geq 0. \tag{2.2}$$

By the well-known basic composition formula (Karlin, 1968 page 17), the left side of Eq. (2.2) is equal to

$$\iint_{u_1 < u_2} \begin{vmatrix} g(t_2 - u_1) & g(t_2 - u_2) \\ g(t_1 - u_1) & g(t_1 - u_2) \end{vmatrix} \begin{vmatrix} \int_0^\infty e^{-sx} F_{X_2}(u_1 - x) dx & \int_0^\infty e^{-sx} F_{X_1}(u_1 - x) dx \\ \int_0^\infty e^{-sx} F_{X_2}(u_2 - x) dx & \int_0^\infty e^{-sx} F_{X_1}(u_2 - x) dx \end{vmatrix} du_1 du_2.$$