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Estimating most productive scale size with double frontiers data envelopment analysis ☆

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A B S T R A C T

In this paper, most productive scale size (MPSS) for input and output mixes is measured from pessimistic point of view by using pessimistic data envelopment analysis (DEA). It is proved that the decision making unit (DMU) with the maximum pessimistic efficiency represents MPSS. However, the optimistic and the pessimistic measurements may identify different DMU as MPSS. To find the optimal DMU that represents MPSS, a double frontiers approach is developed by using the Hurwicz criterion to integrate both the information on the optimistic and the pessimistic frontiers. Numerical examples are provided to show the applications of the proposed methods in estimating MPSS.

1. Introduction

Data envelopment analysis (DEA), introduced by Charnes et al. (1978), is a methodology for measuring the relative efficiency of decision making units (DMUs) with multiple inputs and multiple outputs by employing mathematical programming. Unlike the traditional parametric method in economics, DEA as a nonparametric method does not require many restrictions on the production technology. Furthermore, it is based on the technological aspects of the production correspondences and is not dependent on the estimates of input and output prices. Banker (1984) and Banker et al. (1984) constructed a link between DEA and the estimation of efficient production frontiers with an axiomatic framework.

The contribution of previous research efforts on DEA can be easily found in the literature. For instance, the bibliography of Tavares (2002) lists more than 3000 references from 1978 to 2001, 1200 of these are published in good quality journals. Later an update by Gattoufi et al. (2004) cites more than 1800 works in more than 490 journals from 1951 to 2001. The main methodological development during the past thirty years in DEA was investigated by Cook and Seiford (2009). There is no doubt that DEA is used widely around the world.

An important topic in DEA that links to the returns-to-scale (RTS) is the most productive scale size (MPSS). The concept of the MPSS was introduced into DEA by Banker (1984). Later, Cooper et al. (1996) provided a fractional objective function model for determining the MPSS. Jahanshahloo and Khodabakhshi (2003) proposed an input–output orientation model for estimating the MPSS with a linear objective function. Banker et al. (2004) reviewed the development of MPSS as one part of the literature review of RTS. Recently, Khodabakhshi (2009) discussed the estimation of the MPSS when the stochastic data are obtained.

However, all the papers about the MPSS in DEA are measured from the optimistic point of view. Since the performances of decision making units (DMUs) can also be measured from the pessimistic point of view. Therefore, things may be interesting if one examines the MPSS on production frontier from the pessimistic viewpoint. The literature about the pessimistic measurement in DEA can be found in Wang et al. (2007), Wang and Chin (2007, 2009), Wang and Lan (2011).

The purpose of this paper is to examine the MPSS by using a double frontiers approach. The rest of the paper is constructed as follows. Section 2 briefly reviews the optimistic measurement of the MPSS in DEA. Estimating the MPSS under the pessimistic DEA models is discussed in Section 3. A double frontier approach which integrates both the optimistic measurement and the pessimistic measurement is provided in Section 4. Two numerical examples with the difference among the optimistic measurement, the pessimistic measurement and the double frontiers measurement of the MPSS are presented in Section 5. Finally, the paper concludes in Section 6.

2. MPSS under the optimistic measurement

Suppose we have $j = 1, 2,\ldots, n$ DMUs as (X_j, Y_j), where $X_j = (x_{1j}, x_{2j}, \ldots, x_{mj})$ is a vector of observed inputs and $Y_j = (y_{1j}, y_{2j},\ldots, y_{sj})$ is a vector of observed outputs for DMU_j. Each DMU_j used for efficiency

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comparisons is assumed to have used the same inputs and produced the same outputs.

Definition 1. (Banker, 1984). The production possibility set T can be defined as follows:

$$T = \{(X, Y) | Y \geq 0 \text{ can be produced from } X \geq 0\}.$$

In order to construct an axiomatic production economics framework for relative efficiency measurement in multiple output production settings, Banker et al. (1984) proposed the following four Postulates:

Postulate 1. *Convexity*. If $(X_j, Y_j) \in T$, $j = 1, 2, \ldots, n$, and $\lambda_j \geq 0$ are nonnegative scalars such that $\sum_{j=1}^{n} \lambda_j = 1$, then $\left(\sum_{j=1}^{n} \lambda_j X_j, \sum_{j=1}^{n} \lambda_j Y_j\right) \in T$.
Postulate 2. *Inefficiency Postulate*. (a) If $(X, Y) \in T$ and $\overline{X} \geq X$, then $(\overline{X}, Y) \in T$. (b) If $(X, Y) \in T$ and $\overline{Y} \leq Y$, then $(X, \overline{Y}) \in T$.
Postulate 3. *Ray Unboundedness*. If $(X, Y) \in T$ then $(kX, kY) \in T$ for any $k > 0$.
Postulate 4. *Minimum Extrapolation*. T is the intersection set of all $\hat{T}$ satisfying Postulates 1, 2 and 3 and subject to the condition that each of the observed vectors $\left(\hat{X}, \hat{Y}\right) \in T, j = 1, 2, \ldots, n$.

Denote the DMU under evaluation as DMU_o, where $o \in \{j = 1, 2, \ldots, n\}$. To see its economic meaning of MPSS, we consider the proportions represented by β, α, in $(\beta X_o, \alpha Y_o)$ with $\beta, \alpha \geq 0$ being scalars and X_o and Y_o being input and output vectors, respectively, for DMU_o. With reference to some production possibility set T, a DMU_o with the combination of $(X_o, Y_o) \in T$ is a MPSS if there exists $(\beta X_o, \alpha Y_o)$ for any (β, α) with $\alpha \leq \beta$. A formally definition can be expressed as follows:

Definition 2. (Banker and Thrall, 1992). A production possibility $(X_o, Y_o) \in T$ represents a most productive scale size (MPSS) if and only if for all production possibilities $(\beta X_o, \alpha Y_o) \in T$, we have $\alpha \leq \beta$.

To see the relationship between the optimistic efficiency and the MPSS, we employ the following linear programming:

$$\begin{aligned} &\text{Minimize} \quad \theta, \\ &\text{subject to} \quad \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta x_{io}, \quad i = 1, 2, \ldots, m, \\ &\qquad\qquad\quad \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro}, \quad r = 1, 2, \ldots, s, \\ &\qquad\qquad\quad \lambda_j \geq 0, \quad j = 1, 2, \ldots, n. \end{aligned} \tag{1}$$

The above linear program is known as the CCR model which was proposed by Charnes et al. (1978). The following Proposition identifies the relationship between the efficiency under model (1) and the MPSS.

Proposition 1. (Banker, 1984). *The CCR efficiency rating θ_o^* in the optimal solution to model (1), for a DMU_o is equal to one if and only if it represents MPSS.*

The proof of this Proposition can be found in Banker (1984, pp.39).

Thanks for Banker (1984) work, the relationship between the MPSS and the RTS was constructed as follows

Proposition 2. (Banker, 1984). *If a production possibility $(X_o, Y_o) \in T$ represents a MPSS for the input and output mixes represented by the vectors X_o and Y_o respectively, and if (X_o, Y_o) is neither the smallest nor the largest production possibility for these input and output mixes, then the production correspondence exhibits non-decreasing returns to scale at production possibilities a little smaller than (X_o, Y_o) and non-increasing returns to scale at production possibilities a little larger than (X_o, Y_o). Further, constant returns to scale prevail at (X_o, Y_o).*

The proof of this Proposition can also be found in Banker (1984, pp.37).

3. MPSS under the pessimistic measurement

The efficiencies measured from the pessimistic viewpoint are referred to as the worst relative efficiencies or pessimistic efficiencies. The pessimistic efficiency of DMU_o relative to the other DMUs is measured by the following pessimistic DEA model (Wang et al., 2007):

$$\begin{aligned} &\text{Minimize} \quad \varphi = \sum_{r=1}^{s} \mu_r y_{ro}, \\ &\text{subject to} \quad \sum_{i=1}^{m} \nu_i x_{io} = 1, \\ &\qquad\qquad\quad \sum_{r=1}^{s} \mu_r y_{rj} - \sum_{i=1}^{m} \nu_i x_{ij} \geq 0, \quad j = 1, 2, \ldots, n, \\ &\qquad\qquad\quad \mu_r, \nu_i \geq 0, \quad r = 1, 2, \ldots, s; i = 1, 2, \ldots, m, \end{aligned} \tag{2}$$

where μ_r and ν_i are non-negative weights. The dual model of Eq. (2) can be written as:

$$\begin{aligned} &\text{Maximize} \quad \varphi, \\ &\text{subject to} \quad \sum_{j=1}^{n} \lambda_j x_{ij} \geq \varphi x_{io}, \quad i = 1, 2, \ldots, m, \\ &\qquad\qquad\quad \sum_{j=1}^{n} \lambda_j y_{rj} \leq y_{ro}, \quad r = 1, 2, \ldots, s, \\ &\qquad\qquad\quad \lambda_j \geq 0, \quad j = 1, 2, \ldots, n. \end{aligned} \tag{3}$$

We call the above models, Eqs. (2) and (3), as pessimistic CCR models. They differ from the well-known CCR model (1) in that they minimize the efficiency of DMU_o relative to the others within the range of no less than one, whereas the latter maximizes the efficiency of DMU_o within the range of zero and one.

From Definitions 1 and 2 as well as Propositions 1 and 2, we obtain the following Proposition for identifying the MPSS under model (3).

Proposition 3. *An input–output bundle (X_o, Y_o) is represented as the most productive scale size (MPSS) under model (3) if and only if it attains the maximum optimal value of the objective function among the other input–output bundles.*

Proof. Suppose that the maximum value of the pessimistic CCR model (3) is φ_o^*, and the input–output combination is (X_o, Y_o). We need to show that $\varphi_o^* = \max_{1 \leq j \leq n}\{\varphi_j^*\}$ if and only if (X_o, Y_o) is present in the MPSS.

Now, assume that (X_o, Y_o) is not a MPSS. Then, by Definition 2, there exists (β, α) satisfying $\alpha > \beta$ such that $(\beta X_o, \alpha Y_o)$ is in the production possibility set. Define $X_k = \beta X_o$ and $Y_k = \alpha Y_o$. Because (X_k, Y_k) is in the production possibility set, therefore if we solve model (3) for the input–output combination (X_k, Y_k), we can obtain the optimal value φ_k^*. Since $\varphi_o^* = \max_{1 \leq j \leq n}\{\varphi_j^*\}$, therefore we have $\varphi_k^* < \varphi_o^*$.

Because (X_o, Y_o) and (X_k, Y_k) are both in the production possibility set, thus from model (3), we have

$$\sum_{j=1}^{n} \lambda_j X_j \geq \varphi_o X_o; \quad \sum_{j=1}^{n} \lambda_j Y_j \leq Y_o; \quad \lambda_j \geq 0. \tag{4}$$

Since $X_k = \beta X_o$ and $Y_k = \alpha Y_o$, then $X_o = \frac{X_k}{\beta}$ and $Y_o = \frac{Y_k}{\alpha}$. Therefore we can rewrite Eq. (4) as follows:

$$\sum_{j=1}^{n} \lambda_j X_j \geq \varphi_o \frac{X_k}{\beta}; \quad \sum_{j=1}^{n} \lambda_j Y_j \leq \frac{Y_k}{\alpha}; \quad \lambda_j \geq 0. \tag{5}$$

Let $\omega_j = \alpha \lambda_j$.