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Applying the Model Order Reduction method to a European option pricing model

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ABSTRACT

This paper presents a European option pricing model by applying the Model-Order-Reduction (MOR) method. A European option pricing theorem based on Black–Scholes' equation is implemented by the Finite-Difference Method (FDM). However, the numerical models generated by the FDM could be simplified through the MOR technique, which is based on the concept of an Arnoldi-based Model-Order Reduction algorithm. In terms of computational cost, the MOR models are at least 2 orders of magnitude faster than the original FDM models with a negligible compromise in accuracy.

1. Introduction

An option is a contract giving the buyer a right, but not the obligation, to buy or sell an underlying asset at a specific price on or before a certain date and is also a binding contract with strictly defined terms and properties. A call (put) gives the holder a right to buy (sell) an asset at a certain price within a specific period of time. A put option is very similar to having a short position on a stock (Hunt and Kennedy, 2004). If an option can be exercised on the maturity date only, then the option is a European option. In this paper we only construct a numerical model for a European call option, and in the case of a European put option we can calculate it by using put–call parity.

Over the last few decades, there is an important tendency towards interdisciplinary integration in both academic and practitioner areas, for example, neural networks and fuzzy logic, have been extensively applied in the field of derivatives' pricing (Grudnitski and Osburn, 1993; Hamid and Iqbal, 2004; Hutchinson et al., 1994). These artificial intelligence methods are able to solve several classes of problems that are difficult and sometimes impossible. This study then focuses on the developments of a method for creating European option pricing reduced-order models from FDM (Finite-Difference Method) models. The purpose of this study is to provide a compact as well as an accurate model for pricing European options. The basic idea is to apply an Arnoldi-based Model Order Reduction (MOR) technique [1, 2, 3, 4] on the system matrices formulated by the FDM discretization processes from the Black–Scholes equation (Wang and White, 1998; Yu et al., 2003).[1] The MOR technique is a popular research topic in EDA (electronics design automation) industries, which speeds up computation and reduces storage requirements by replacing a large-scale system of differential or difference equations by one of substantially lower dimension that has nearly the same response characteristics (Yang and Yen, 2005; Yang and Yu, 2004).

This paper is organized as follows. Section 2 describes the governing equations (Black–Scholes equation) for European option pricing and the boundary conditions. The algorithm of MOR is presented in Section 3. Section 4 presents the simulated cases of a European option pricing model. Comparisons between the results by the full-meshed FDM model and the MOR models are then demonstrated and discussed. Section 5 concludes this paper.

2. European option pricing (Black–Scholes equation)

2.1. Black–Scholes (1973) equation

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t \\ dB_t &= rB_t dt, \end{aligned} \tag{1}$$

[1] Model Order Reduction (MOR) is a branch of systems and control theory, which studies properties of dynamical systems in application for reducing their complexity, while preserving (to the possible extent) their input–output behavior. (argued by *Model Order Reduction site at MIT*).

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where S_t is the stock price, μ is the drift term of the stock process, σ is the volatility of the stock price, W_t is the standard Brownian motion, B_t is the bond price, and r is the interest rate.

$$V_t = h_S S_t + h_B B_t \tag{2}$$

By self-financing assumption [5]:

$$\begin{aligned} dV_t &= h_S dS_t + h_B dB_t \\ &= h_S(\mu S_t dt + \sigma S_t dW_t) + h_B(rB_t dt) \\ &= (h_S \mu S_t + h_B r B_t)dt + h_S \sigma S_t dW_t. \end{aligned} \tag{3}$$

If we assume that the hedging portfolio's price is the function of t and S_t, then:

$$V_t = V_t(t, S_t)$$

By the Ito lemma:

$$\begin{aligned} dV_t &= \frac{\partial V_t}{\partial t}dt + \frac{\partial V_t}{\partial S_t}dS_t + \frac{1}{2}\frac{\partial^2 V_t}{\partial S_t^2}dS_t dS_t \\ &= \frac{\partial V_t}{\partial t}dt + \frac{\partial V_t}{\partial S_t}(\mu S_t dt + \sigma S_t dW_t) + \frac{1}{2}\frac{\partial^2 V_t}{\partial S_t^2}\sigma^2 S_t^2 dt \\ &= \left(\frac{\partial V_t}{\partial t} + \frac{\partial V_t}{\partial S_t}\mu S_t + \frac{1}{2}\frac{\partial^2 V_t}{\partial S_t^2}\sigma^2 S_t^2\right)dt + \frac{\partial V_t}{\partial S_t}\sigma S_t dW_t. \end{aligned} \tag{4}$$

Coefficient matching between Eqs. (3) and (4) results in:

$$\begin{aligned} h_S &= \frac{\partial V_t}{\partial S_t} \\ h_B &= \frac{1}{rB_t}\left(\frac{\partial V_t}{\partial t} + \frac{1}{2}\frac{\partial^2 V_t}{\partial S_t^2}\sigma^2 S_t^2\right). \end{aligned} \tag{5}$$

Substituting Eq. (5) into Eq. (2), we therefore get the standard Black–Scholes PDE as below:

$$V_t = \frac{\partial V_t}{\partial S_t}S_t + \frac{1}{r}\left(\frac{\partial V_t}{\partial t} + \frac{1}{2}\frac{\partial^2 V_t}{\partial S_t^2}\sigma^2 S_t^2\right). \tag{6}$$

For a European call option with maturity T, we also have a boundary condition:

$$V_t(T, S_T) = \max(0, S_T - K), \tag{7}$$

where K is the strike price of the call option.

2.2. Transform to diffusion equation

To consider the Black–Scholes Eq. (6), we apply the following change of variables:

$$\begin{aligned} &x = \ln\left(\frac{S_t}{K}\right) \text{ and } \tau = (T-t)\left(\frac{1}{2}\sigma^2\right) \\ &u(x,\tau) = e^{\gamma x + (\gamma^2 + k)\tau}\frac{V_t}{K}, \end{aligned} \tag{8}$$

where $\gamma = \frac{1}{2}(k-1)$ and $k = 2r/\sigma^2$. Next, we have:

$$\frac{du}{d\tau} = \frac{d^2 u}{dx^2}, \tag{9}$$

with the "u vs. τ" or "u vs. x" curves calculated by the FDM code and the MOR techniques.

2.3. Boundary condition

The points of interest in the interval $[x_{\min}, x_{\max}]$ are the boundary points between $x_{\min}$ and $x_{\max}$. Initially, at $\tau = 0$, we know $u(x,\tau)$ equals the payoff function $u(x,0)$. However, at time points $\tau > 0$, we have to choose boundary conditions to represent the behavior of the function $u(x,\tau)$ at $x \to \pm\infty$.

From put–call parity $\max(S_T - K,0) - \max(K - S_T,0) = S_T - K$, we deduce that the put option is equivalent to a short position on a forward contract and a long position on a call option with the same parameters K and T. At time t before expiration, we must then have:

$$P(t) = Ke^{-r(T-t)} - S_t + C(t), \tag{10}$$

where $P(t)$ is the price of the put option at time t, and $C(t)$ is the price of the call option at time t.

Consider the case when $S_t \to 0$. The price of a call option goes to zero:

$$C_{\min} = 0. \tag{11}$$

As $S_t \to \infty$, the value of a put option goes to zero:

$$P_{\max} = 0. \tag{12}$$

The value of a call option by the put–call parity is again:

$$C_{\max} = S_t - Ke^{-r(T-t)}. \tag{13}$$

We have to transform these boundary conditions into terms of (x,τ). The transformation is in Eq. (8). Therefore, the call boundary condition of Eq. (13) is equivalent to:

$$u_{\sup}(\tau) = e^{rx_{\max} + (\gamma^2 + k)\tau}\left(e^{x_{\max}} - e^{-k\tau}\right). \tag{14}$$

To summarize, for the call option the boundary conditions are:

$$\begin{aligned} &u_{\inf}(\tau) = 0 \\ &u_{\sup}(\tau) = e^{rx_{\max} + (\gamma^2 + k)\tau}\left(e^{x_{\max}} - e^{-k\tau}\right). \end{aligned} \tag{15}$$

3. Methodology of Model Order Reduction (MOR)

The FDM solver is capable of calculating European options, however, the computational cost is very expensive if there is a large number of nodes in the computational domain. Fortunately, the governing equation as well as the boundary conditions, as shown in Eq. (9) which is a linear equation, and the system matrices generated by the FDM approximation process for Eq. (9) could be reduced by an Arnoldi-based Model Order Reduction technique (Marques et al., 2004; Wang and White, 1998; Yu et al., 2003). The computation time of MOR is negligible and the orders of magnitude are faster than any traditional discretization schemes such as fine-mesh FDM. A detailed description of the Model Order Reduction process is as follows.

The dynamic system equation formulated by the FDM approximation of governing equation (Eq. (9)) and the boundary conditions (Eq. (15)) is written as:

$$\begin{aligned} \dot{\vec{x}} &= \underline{\underline{A}}\vec{x} + \underline{\underline{B}}\vec{u}, \\ \vec{y} &= \underline{\underline{C}}^T\vec{x} + \underline{\underline{D}}\vec{u} \end{aligned} \tag{16}$$

where $\underline{\underline{A}}$ is an $n \times n$ matrix, n is the total number of nodes, $\vec{x}$ is the vector which contains the unknown temperature distribution on each node,