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Dependence of defaults and recoveries in structural credit risk models

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ABSTRACT

The current research on credit risk is primarily focused on modelling default probabilities. Recovery rates are often treated as an afterthought; they are modelled independently, in many cases they are even assumed to be constant. This despite their pronounced effect on the tail of the loss distribution. Here, we take a step back, historically, and start again from the Merton model, where defaults and recoveries are both determined by an underlying process. Hence, they are intrinsically connected. For the diffusion process, we can derive the functional relation between expected recovery rate and default probability. This relation depends on a single parameter only. In Monte Carlo simulations we find that the same functional dependence also holds for jump-diffusion and GARCH processes. We discuss how to incorporate this *structural recovery rate* into reduced-form models, in order to restore essential structural information which is usually neglected in the reduced-form approach.

1. Introduction

An accurate description of portfolio credit risk is of vital interest for any financial institution. It is also a prerequisite for realistic ratings of structured credit derivative products. Furthermore, it is a crucial aspect in banking regulations. We can distinguish two conceptually different approaches to credit risk modelling: structural and reduced-form approaches. The structural models go back to Black and Scholes (1973) and Merton (1974). The Merton model assumes that a company has a certain amount of zero-coupon debt which becomes due at a fixed maturity date. The market value of the company is modelled by a stochastic process. A possible default and the associated recovery rate are determined directly from this market value at maturity. In the reduced-form approach default probabilities and recovery rates are described independently by stochastic models. The aim is to describe the dependence of these quantities on common (macroeconomic) covariates or risk factors. For some well known reduced-form model approaches see, e.g., Duffie and Singleton (1999), Hull and White (2000), Jarrow and Turnbull (1995), Jarrow et al. (1997) and Schönbucher (2003). First Passage Models constitute a third approach—which is usually regarded as structural, but is better described as a mixed approach. They were first introduced by Black and Cox (1976). As in the Merton model, the market value of a company is modelled by a stochastic process. Default of a company occurs as soon as the market value falls below a certain threshold. In contrast to the Merton model, default can occur at any time prior to default. In this approach, the recovery rate is not determined by the underlying process for the market value. Instead recovery rates are modelled independently, for example, by a reduced-form approach (see e.g., Asvanunt and Staal (2009a,b)). In some cases recovery rates are even assumed to be constant, for instance, in Giesecke (2004). The independent modelling of default and recovery rates leads to a serious underestimation of large losses.

In this paper we take a step back, historically, and revisit the Merton model, where defaults and recoveries are both determined by an underlying process. Hence, they are intrinsically connected. For a correlated diffusion process the Merton model has been treated analytically, e.g., in Bluhm et al. (2002) and in Giesecke (2004). In a straightforward calculation we can also derive the functional relation between expected recovery rate and default probability. This relation depends on a single parameter only. In our Monte Carlo simulations we find that the same functional dependence also holds for other processes like jump-diffusion and GARCH. We discuss how to incorporate this relation into reduced-form models, in order to restore essential structural information which is usually neglected in the reduced-form approach.

The paper is organised as follows. We give a short introduction to the Merton model in Section 2. In Section 3 we treat the diffusion case analytically and compare the results to Monte Carlo simulations. In Sections 4 and 5 we extend the Merton model to jump-diffusion and

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GARCH processes. For both processes we find the same functional dependence between default and recovery rates as in the diffusion case. We discuss the applicability of the structural recovery rate beyond the Merton model in Section 7. We summarise our findings in Section 7.

2. Merton model

The Merton model is based on the assumption that a company k has a certain amount of zero-coupon debt; this debt has the face value F_k and becomes due at maturity time T. The company defaults if the value of its assets at time T is less than the face value, i.e., if $V_k(T)<F_k$. The recovery rate then reads $R_k=V_k(T)/F_k$ and the loss given default is

$$L_k^* = 1-R_k = \frac{F_k-V_k(T)}{F_k}. \tag{1}$$

We denote the loss given default with an asterisk to distinguish it from the loss including non-default events. The individual loss can be expressed as

$$L_k = \left(1-\frac{V_k(T)}{F_k}\right)\Theta\left(1-\frac{V_k(T)}{F_k}\right), \tag{2}$$

where Θ is the Heaviside function. In the Merton model, defaults and losses—and hence also recoveries—are directly determined by the asset value at maturity. Therefore, stochastic modelling of the market value $V_k(t)$ of a company allows to assess its credit risk. Let $p_{V_k}(V_k(T))$ be the probability density function (pdf) of the market value at maturity. Then the default probability is given by

$$P_{\mathrm{D},k} = \int_0^{F_k} p_{V_k}(V_k(T))\mathrm{d}V_k(T) \tag{3}$$

and the expected recovery rate can be calculated as

$$\langle R_k\rangle = \frac{1}{P_{\mathrm{D},k}}\int_0^{F_k}\frac{V_k(T)}{F_k}p_{V_k}(V_k(T))\mathrm{d}V_k(T). \tag{4}$$

Let us now consider a portfolio of K credit contracts, where the market value of each company k is correlated to one or more covariates. Conditioned on the realisations of the covariates we obtain different values for $P_{\mathrm{D},k}$ and $\langle R_k\rangle$. In fact, we find a functional dependence between default probability and recovery rate. This is in stark contrast to modelling approaches which assume an independence of these quantities.

3. Correlated diffusion

3.1. Analytical discussion

Assuming an underlying diffusion process for the company value we can easily derive all results analytically. To keep the notation simple we consider a homogeneous portfolio of size K with the same parameters for each asset process, and with the same face value, $F_k=F$, and initial market value, $V_k(0)=V_0$. We model the time evolution of the market value of a single company k by a stochastic differential equation of the form

$$\frac{\mathrm{d}V_k}{V_k} = \mu\mathrm{d}t + \sqrt{c}\sigma\ \mathrm{d}W_\mathrm{m} + \sqrt{1-c}\sigma\ \mathrm{d}W_k. \tag{5}$$

This describes a correlated diffusion process with a deterministic term μdt and a linearly correlated diffusion. The parameters of this process are the drift constant μ, the volatility σ and the correlation coefficient c. The Wiener processes denoted by $\mathrm{d}W_k$ and $\mathrm{d}W_\mathrm{m}$ describe the idiosyncratic and the market fluctuations, respectively.

For discrete time increments $\Delta t=T/N$, where the time to maturity T is divided into N steps, we arrive at the discrete formulation of Eq. (5). The market value of company k at maturity can be written as

$$V_k(T) = V_0\prod_{t=1}^{N}\left(1+\mu\Delta t+\sqrt{c}\sigma\eta_{\mathrm{m},t}\sqrt{\Delta t}+\sqrt{1-c}\sigma\varepsilon_{k,t}\sqrt{\Delta t}\right). \tag{6}$$

We define the market return X_m as the average return of all single companies k over the time horizon up to maturity,

$$X_\mathrm{m} = \frac{1}{K}\sum_{k=1}^{K}\left(\frac{V_k(T)}{V_0}-1\right) \tag{7}$$

$$=\frac{1}{K}\sum_{k=1}^{K}\prod_{t=1}^{N}\left(1+\mu\Delta t+\sqrt{c}\sigma\eta_{\mathrm{m},t}\sqrt{\Delta t}+\sqrt{1-c}\sigma\varepsilon_{k,t}\sqrt{\Delta t}\right)-1. \tag{8}$$

For $K\to\infty$ we can express the average over k as the expectation value of $\varepsilon_{k,t}$. Due to the independence of $\varepsilon_{k,t}$ for different k and t, we can write

$$X_\mathrm{m}+1 = \prod_{t=1}^{N}\left(1+\mu\Delta t+\sqrt{c}\sigma\eta_{\mathrm{m},t}\sqrt{\Delta t}+\sqrt{1-c}\sigma\left\langle\varepsilon_{k,t}\right\rangle\sqrt{\Delta t}\right), \tag{9}$$

with

$$\left\langle\varepsilon_{k,t}\right\rangle = 0. \tag{10}$$

Thus, expression (9) simplifies to

$$\begin{aligned} X_\mathrm{m}+1 &= \prod_{t=1}^{N}\left(1+\mu\Delta t+\sqrt{c}\sigma\eta_{\mathrm{m},t}\sqrt{\Delta t}\right) \\ &= \exp\left(\sum_{t=1}^{N} ln\left(1+\mu\Delta t+\sqrt{c}\sigma\eta_{\mathrm{m},t}\sqrt{\Delta t}\right)\right) \\ &\approx \exp\left(\left(\mu-\frac{c\sigma^2}{2}\right)T+\sigma\sqrt{c\Delta t}\sum_{t=1}^{N}\eta_{\mathrm{m},t}\right). \end{aligned} \tag{11}$$

In the last step of the calculation we expanded the logarithm up to first order in Δt. The random variables $\eta_{\mathrm{m},t}$ are standard normal distributed. Therefore the variable

$$\ln(X_\mathrm{m}+1) = \left(\mu-\frac{c\sigma^2}{2}\right)T+\sigma\sqrt{cT}\frac{1}{\sqrt{N}}\sum_{t=1}^{N}\eta_{\mathrm{m},t} \tag{12}$$

is normal distributed with mean $\mu T-\frac{1}{2}c\sigma^2T$ and variance $c\sigma^2T$. This implies a shifted log-normal distribution for the market return itself,

$$p_{X_\mathrm{m}}(X_\mathrm{m}) = \frac{1}{(X_\mathrm{m}+1)\sqrt{2\pi c\sigma^2T}}\exp\left(-\frac{\left(\ln(X_\mathrm{m}+1)-\mu T+\frac{1}{2}c\sigma^2T\right)^2}{2c\sigma^2T}\right). \tag{13}$$

For a single company k we can write

$$\ln\frac{V_k(T)}{V_0} = \sum_{t=1}^{N}\ln\left(1+\mu\Delta t+\sqrt{c}\sigma\eta_{\mathrm{m},t}\sqrt{\Delta t}+\sqrt{1-c}\sigma\varepsilon_{k,t}\sqrt{\Delta t}\right) \tag{14}$$

$$\approx \ln(X_\mathrm{m}+1)-\frac{(1-c)\sigma^2}{2}T+\sigma\sqrt{(1-c)T}\frac{1}{\sqrt{N}}\sum_{t=1}^{N}\varepsilon_{k,t}. \tag{15}$$

Conditioned on a fixed value for the market return X_m, all variables $V_k(T)$ are independent and $lnV_k(T)/V_0$ is normal distributed with mean $ln(X_\mathrm{m}+1)-\frac{1}{2}(1-c)\sigma^2T$ and variance $(1-c)\sigma^2T$. Since we consider a