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A note on the use of fractional Brownian motion for financial modeling

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ABSTRACT

In the second part of the past decade, the usage of fractional Brownian motion for financial models was stuck. The favorable time-series properties of fractional Brownian motion exhibiting long-range dependence came along with an apparently insuperable shortcoming: the existence of arbitrage. Within the last two years, several new models using fractional Brownian motion have been published. However, still the problem remains unsolved whether such models are reasonable choices from an economic perspective.

In this article, we take on a straightforward mathematical argument in order to clarify when and why fractional Brownian motion is suited for economic modeling: We provide a fractional analog to the work of Sethi and Lehoczky (1981) thereby confirming that fractional Brownian motion and continuous tradability are incompatible. In the light of a market microstructure perspective to fractional Brownian motion, it becomes clear that the correct usage of fractional Brownian motion inherently implies dynamic market incompleteness.

Building a bridge to application, we show that one peculiar, but nevertheless popular result in the literature of fractional option pricing can be well explained by the fact that authors disobeyed this need for compatibility.

1. Introduction

Fractional Brownian motion was introduced by Mandelbrot and van Ness (1968). Being extensions of classical Brownian motion, both models of randomness still have some key properties in common, most importantly, they are Gaussian. For all Hurst parameters $H \neq \frac{1}{2}$, there are however also important differences: while Brownian motion has independent increments, the increments of fractional Brownian are serially correlated. Thereby, new information has a persisting influence on the process, which implies a certain level of predictability: In contrast to the classical Brownian case, the historical trajectory of the process does matter when forecasting its future evolution.

Our analysis will be based on a continuous time market setup with two assets. We introduce a riskless asset A_t following

$$dA_t = rA_t dt, \tag{1}$$

and, referring to the definition of fractional Brownian motion, a risky asset S_t by means of a geometric fractional Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dB_t^H. \tag{2}$$

The parameters for the riskless interest rate r as well as for the drift μ and the volatility σ of the stock price process are constant. The differential Eq. (2) can be interpreted in different ways depending on the chosen stochastic integration calculus. Throughout this paper we will focus on pathwise integration on one hand and Wick-based integration on the other hand.

The remainder of the paper is organized as follows: In the next section, we briefly recall the debate concerning the problem of arbitrage that is inherent to markets driven by fractional Brownian motion. We also look at the recent literature focusing on a market microstructure perspective toward fractional Brownian motion. Keeping these results in mind, we use in Section 3 a fractional analog to the work of Sethi and Lehoczky (1981) to show that fractional Brownian motion and continuous tradability are incompatible. In Section 4, we then show how the conceptual discussion can be brought to a more applied basis showing that the correct handling of this incompatibility leads to results that are more intuitive. In Section 5 we summarize our main findings and revisit them based on the insights gained by the market microstructure perspective.

2. Characteristics of financial models with fBM

Within the last two years, a number of articles have been published choosing fractional Brownian motion as an underlying diffusive process (e.g. Gu et al., 2012; Meng and Wang, 2010; Xiao et al., 2010). The references used therein seem to show that some of the insights of the first part of the last decade have been buried in oblivion. This may be due to the fact, that the discussion then had become rather technical. Still, a clear statement whether to use fractional Brownian motion as a model in finance or not seems to be overdue.

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Since Rogers (1997), there has been an ongoing discussion about the usage of fractional Brownian motion within financial models. While most of the literature focuses on arbitrage and its exclusion, some publications also discuss market microstructure foundations of fractional Brownian motion.

The predictability mentioned above causes problems when modeling stock prices by fractional Brownian motion. Rogers (1997) derived arbitrage possibilities in a fractional Bachelier type model and stated fractional Brownian motion to be an unsuitable candidate for usage in financial models. However, the question of generality concerning his results attracted some further discussion as the findings of Rogers (1997) were limited to the case of a restricted linear setting without drift. Shiryayev (1998) constructed an explicit arbitrage strategy within the fractional market setting given by Eqs. (1) and (2) based on pathwise integrals. Similar results were derived by Bender (2003) and Dasgupta and Kallianpur (2000).

While these first results concerning financial market models based on fractional Brownian motion looked rather disillusioning, it was still hoped to remedy the shortcomings of the suggested market setting. The research interest in this field was re-encouraged by new insights in stochastic analysis mainly initiated by the work of Duncan et al. (2000) who provided a stochastic integration calculus with respect to fractional Brownian motion based on the Wick product. This integration concept makes it possible to draw parallels to the well-known Itô calculus.

Despite this innovative stochastic integration concept, things did not really change for the better. Several years before, Delbaen and Schachermayer (1994) had proved a powerful result holding in general for continuous time market models: Irrespective of the choice of integration theory, a weak form of arbitrage called free lunch with vanishing risk can be excluded if and only if the underlying stock price process S is a semimartingale. It is easy to verify that, due to their persistent character, processes driven by fractional Brownian motion are not semimartingales. Furthermore, Cheridito (2003) succeeded in constructing explicit arbitrage strategies both in the fractional Bachelier model and in the fractional Black–Scholes market no matter which kind of integration method—pathwise or Wick-based calculus—was used.

This critique is well-justified as long as underlying concepts like absence of arbitrage and the self-financing property are interpreted in their traditional sense. However, modifications in these definitions have been proposed, among them the approaches due to Hu and Øksendal (2003) and Elliott and van der Hoek (2003). They implemented the Wick product into the definitions of the portfolio value and/or the property of being self-financing.

Though, the denaturalization of the Wick based concept from its assignment as part of an integration concept led to intense discussions. Actually, the seemingly encouraging result of a fractional Black–Scholes market excluding arbitrage provided by Hu and Øksendal (2003) or Elliott and van der Hoek (2003) entailed further models based thereon (e.g. Benth, 2003; Della Ratta et al., 2008; Necula, 2002). However, although this pricing approach grew in popularity, some serious concerns questioned the usage of Wick products beyond pure integration theory. The suitability of Wick-based definitions of fundamental economic concepts was firstly doubted by Sottinen and Valkeila (2003). The irritating results that such an extension of the Wick product can cause, were later summarized by Bjork and Hult (2005). The most striking one is that the definition of the value process introduced by Hu and Øksendal (2003) contradicts economic intuition: Even the knowledge of the current realization of the stock price process and the position held in stocks would not be sufficient to calculate the realization of the portfolio value. Also the modified self-financing condition yields bizarre results. Bjork and Hult (2005) note that if one would follow the idea of this self-financing condition in real economics, "your proposed book value will […] entail a violation of corporate law (and you may be prosecuted)".

Some authors like Rogers (1997) and Cheridito (2001a) suggested approaches with a regularized stochastic stock price process, where the weighting kernel of the integral representation of fractional Brownian motion is replaced. Such a modified stochastic process is now a well-manageable semimartingale but still close to fractional Brownian motion. While this mathematical construction ensures the existence of a unique martingale measure, the problem of how to determine the correct regularization kernel remains unsolved (see Sottinen, 2001).

A different approach where the stochastic process of the stock price is also transformed into a semimartingale was again suggested by Cheridito (2001b): In a mixed model, fractional Brownian motion is endowed with an additional term of classical Brownian motion. The stock price dynamics of the so called mixed fractional Brownian motion are

$$dS_t = \mu S_t dt + \epsilon S_t dB_t + \sigma S_t dB_t^H. \tag{3}$$

In this model one can approximate geometric fractional Brownian motion inserting small values for ϵ. It is even possible to price European call options and the pricing formula heavily resembles the Black–Scholes price of a call, however the volatility becomes σ_ϵ (Cheridito, 2001b). We will have a closer look at the consequences of this result in Section 3.

More recently, some interest has been directed toward a market microstructure foundation of fractional Brownian motion. Klüppelberg and Kühn (2004) introduce a so-called shot noise process. The idea is to model new information that arrives at random times and subsequently diffuses into the market. Thereby, new information may have a long lasting influence on the price process. In the limit, the model of Klüppelberg and Kühn (2004) converges to fractional Brownian motion. Bayraktar et al. (2006) model inert behavior of investors. That means, that after a trading activity, there is a certain probability that an investor does not carry out any transaction during the next period. Explicitly this is done by introducing a stochastic process x_t^a representing the trading mood of the investor a. For all times where x_t^a equals zero, the investor is in an inactive state. Orders of market participants arrive asynchronously and trades are cleared by a market maker who sets prices reacting on the imbalance of demand and supply. In the limit, the price process then tends to a geometric fractional Brownian motion. If investors do not show this inert behavior, a classical Brownian motion is obtained. A market model with both inert investors and continuously trading market participants consequently leads to a mixed model à la Cheridito (2001b).

The idea of inert investors that do not trade continuously has its counterpart in the following approach: Cheridito (2003) proves that if a single investor cannot accomplish two consecutive transactions infinitesimally fast, the market becomes free of arbitrage. However, the market then is dynamically incomplete. In the following section we will provide a detailed motivation of this restriction on tradability.

3. FBM and continuous trading are incompatible

For decreasing values of ϵ Cheridito's mixed model approaches the fractional Brownian motion market. However, with $\epsilon \to 0$, the volatility of the option σ_ϵ also vanishes and hence randomness disappears. Cheridito (2001b) argues that as soon as the Brownian noise component disappears the market participants can (i) exploit the predictability of the fractional Brownian motion, (ii) find an appropriate trading strategy and (iii) eliminate randomness.

This strange behavior in the limiting case above is an immediate consequence of the assumption of continuously trading investors. In this section we derive a fractional analog to the work of Sethi and Lehoczky (1981). The surprising outcome of this analog is that in the fractional context independent of the integration calculus applied continuous hedging eliminates risk and thereby makes option prices deterministic. For further details, see Rostek (2009).