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Multi-period mean–variance asset–liability management with uncontrolled cash flow and uncertain time-horizon

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ABSTRACT

This paper considers an asset–liability management problem under a multi-period mean–variance model with uncontrolled cash flow and uncertain time-horizon. The difference from the existing literature is that the liability is assumed to be influenced not only by the stochastic return of the liability but also by some uncontrolled cash flows, which can be explained as, for example, stochastic expenditure of individual investors, or claim processes of insurers. Firstly, the original problem is translated into a standard multi-period stochastic optimal control problem by introducing a Lagrange multiplier, and the corresponding analytical solution is derived by adopting the dynamic programming approach. Secondly, according to Lagrange duality theorem, closed-form expressions for the efficient investment strategy and the mean–variance efficient frontier are obtained. Moreover, a multi-period version of two-fund separation theorem is proved, and some special cases are discussed. Finally, some numerical examples are presented to illustrate our results.

1. Introduction

By using variance to measure risk, Markowitz (1952) establishes the famous mean–variance (M–V) portfolio selection model, which has become the milestone of modern finance theory. But due to the fact that variance is non-separable in the sense of dynamic programming, M–V portfolio selection model is not extended to the dynamic cases for a long time. Until recently, Li and Ng (2000) and Zhou and Li (2000) use an embedding technique to derive analytical solutions to multi-period and continuous-time M–V models, respectively. After that, many scholars further explore the dynamic M–V portfolio selection problems under various realistic conditions. For example, Zhu et al. (2004) and Bielecki et al. (2005) consider a multi-period and a continuous-time M–V portfolio selection problem with bankruptcy control, respectively. Celikyurt and Özekici (2007) investigate a multi-period M–V portfolio optimization problem in a stochastic market environment where there is finite number of state and the state transition process is a Markov chain. Elliott et al. (2010) study a continuous-time M–V portfolio selection problem under a hidden Markovian regime-switching model. Fu et al. (2010) consider a continuous-time M–V model with different borrowing-lending rates.

Recently, there are two important extensions of the M–V portfolio selection problems. The first one is research on the M–V portfolio selection problems with uncertain time-horizon. In reality, since there are many unpredictable stochastic factors such as unexpected huge consumption, serious illness or sudden death, investors may exit the financial market before the time they planned. To our knowledge, this kind of problem is first studied by Yaari (1965), who considers a consumption, investment and life insurance problem with uncertain lifetime. Pliska and Ye (2007) consider the continuous-time optimal investment consumption, and life-insurance purchase rules for a wage earner whose lifetime is random. Then, under continuous-time expected utility maximization model, Merton (1971) investigates an optimal investment and consumption problem when the exit time follows a Poisson process, and Christophette et al. (2008) consider an optimal investment problem with general uncertain time-horizon. Under static M–V framework, Martellini and Urosevic (2006) study a portfolio selection problem in cases with exogenously and endogenously uncertain exit times. Wu and Li (2011) investigate a multi-period M–V portfolio optimization problem with uncertain exit time and regime switching market environment.

The other important extension of M–V portfolio selection problems is research on the asset–liability management (ALM) problems under M–V criterion. It is well known that ALM problems are of both theoretical interest and practical importance. For example, ALM has extensive applications in banks, pension funds and insurance companies. Meanwhile, there are many scholars focusing on ALM problems. Sharpe and Tint (1990) first consider an ALM problem under the static M–V framework. Based on the multi-period M–V framework, Leippold et al. (2004) study an optimal portfolio selection problem with uncontrolled liability; Yi et al. (2008) and Leippold et al. (2011) extend the work of Leippold et al. (2004) to cases with uncertain exit time and endogenous

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liabilities, respectively; Chen and Yang (2011) investigate an ALM problem with regime switching parameters. In addition, there are also many scholars studying ALM problems under continuous-time M–V framework models, such as Chiu and Li (2006), Xie et al. (2008), Chen et al. (2008), Xie (2009), Li and Shu (2011) and Zeng and Li (2011).

As far as we know, most of the existing literature on multi-period M–V ALM problems follow the assumption imposed by Leippold et al. (2004) that liabilities are only determined by their returns. In reality, however, besides the returns of liabilities, uncontrolled cash flow is another important factor that would affect liabilities directly. For example, claims encountered by the insurers, dividend payments of firms and stochastic expenditure of individual investors, which are uncontrolled cash flow, can influence liabilities.

In this paper, by incorporating the uncontrolled cash flow into the liability dynamic process, we consider a multi-period M–V ALM problem with uncertain time-horizon. In addition, it is well known that the two-fund separation theorem (see Tobin, 1958 for more details) in Markowitz's M–V model is very important in both academia and industry, for example, which is the foundation of the famous capital asset pricing model (CAPM) (see Sharpe, 1964). In this paper, we further investigate the two-fund separation theorem under our model. What is more, since our ALM problem has two state variables, i.e., the wealth and the liability, adding uncontrolled cash flow factor into the liability dynamic process would increase the computational complexity in obtaining closed form solutions to the ALM model. In this paper, we will synthetically adopt the Lagrange duality method and the dynamic programming approach for solving the multi-period M–V ALM problem.

The reminder of this paper is organized as follows: In Section 2, under the multi-period M–V framework, we set up an ALM problem with uncertain time-horizon, where the liability is affected by both the uncontrolled cash flows and the returns of the liabilities. In Section 3, the original problem is translated into a standard multi-period stochastic optimal control problem by introducing a Lagrange multiplier, and the corresponding analytical solution is obtained by the dynamic programming approach. In Section 4, the efficient investment strategy and the efficient frontier for the original M–V model are derived explicitly. Section 5 presents a multi-period version of two-fund separation theorem. Some special cases are discussed in Section 6, and Section 7 provides some numerical examples to illustrate our results. Section 8 concludes this paper. Proof of propositions and theorems are relegated to Appendix A.

2. Model and assumptions

Assume that the financial market consists of $n+1$ assets which can be all risky or include one risk-free asset. Let $e_k=(e_k^0,e_k^1,e_k^2,\ldots,e_k^n)'$ denote the return vector of the $n+1$ assets over period k, $k=0,1,\ldots,T-1$, where e_k^i is the return of the ith asset over period k, and A' denotes the transpose of a matrix or vector A. Consider an investor, who enters the market at time 0 with initial wealth x_0 and initial liability l_0, plans to make investment within T period. However, during the investment process, the investor may be forced to exit the market at some time τ before T for some uncontrollable reasons. Therefore, the actual exit time of the investor is $T\wedge\tau:=min\{T,\tau\}$. Suppose that τ is a positive exogenous random variable with discrete probability function $\tilde{p}_j=\mathbb{P}(\tau=j),(j=1,2,\ldots)$ and distribution function $F(t)=\sum_{j\le t}\tilde{p}_j$, where $\mathbb{P}$ is the probability measure. Therefore, $T\wedge\tau$ has the probability function as follows

$$p_j=\mathbb{P}(T\wedge\tau=j)=\tilde{p}_j, j=1,2,\ldots,T-1;$$

$$p_T=\mathbb{P}(T\wedge\tau=T)=1-F(T-1)=1-\sum_{i=1}^{T-1}\tilde{p}_i.$$

Moreover, we suppose that the investor has an exogenous liability. In the existing literature, the liability dynamic process is only assumed to be (see Chen and Yang, 2011; Leippold et al., 2004; Yi et al., 2008)

$$l_{k+1}=q_kl_k, k=0,1,\ldots,T-1,$$

where q_k is an exogenous random variable representing the stochastic growth rate or return rate of the liability over period k. However, in real-world, there would be a cash inflow or outflow during the investment process. For example, the insurers need to pay for claims; firms distribute extra dividends; and the individual investors have uncertain expenditures. They can either increase or decrease the liabilities. Therefore, in this paper we consider a more general form of liability dynamic process, i.e.,

$$l_{k+1}=q_kl_k+b_k, k=0,1,\ldots,T-1, \tag{1}$$

where q_k and b_k are both exogenous random variables, the meaning of q_k is the same as above, and b_k represents the uncontrolled cash flow which affects the liability. $b_k>0$ means that there is a cash outflow over period k which increases the liability; $b_k<0$ implies that there is a cash inflow over period k which decreases the liability; and $b_k=0$ suggests there is no cash inflow or outflow over period k. To ensure that l_k is indeed liability, we assume that $l_k\ge 0$ almost surely for all $k=0,1,\ldots,T-1$; namely, the exogenous liability is almost surely nonnegative for every period. Notice that if $b_k=0$ at any period k, our liability dynamic process is the same as that in Leippold et al. (2004) and Yi et al. (2008). Therefore, in some sense we extend the models in Leippold et al. (2004) and Yi et al. (2008).

Let x_k and l_k denote the wealth and the liability of the investor at time k, respectively, and u_k^i, $i=1,2,\ldots,n$ denote the amount invested in the ith asset. The amount invested in the 0th asset is $x_k-\sum_{i=1}^n u_k^i$. Moreover, we assume that transactions are carried out at the beginning of every period and no transaction cost or tax is involved, then the dynamics of the investor's wealth process can be described as

$$x_{k+1}=x_ke_k^0+P_k'u_k, k=0,1,\ldots,T-1, \tag{2}$$

where $P_k=(e_k^1-e_k^0,e_k^2-e_k^0,\ldots,e_k^n-e_k^0)'$, and $u_k=(u_k^1,u_k^2,\ldots,u_k^n)'$. Similar to most of the existing literature, we make some assumptions as follows:

Assumption 1. $E[e_ke_k']$ is positive definite for $k=0,1,\ldots,T-1$.

Assumption 2. Random series $\Upsilon_k=(P_k,p_k,b_k)$, $k=0,1,\ldots,T-1$, are statistically independent, i.e., Υ_i and Υ_j are independent for $i,j=0,1,\ldots,T-1$ if $i\neq j$.

Assumption 3. $p_T>0$, namely, the probability of exit the market at time T is positive.

Assumption 4. $E[P_k]\neq\vec{0}$, $k=0,1,\ldots,T-1$, where $\vec{0}$ is an n-dimension zero vector.

Throughout this paper, let $\mathcal{F}_k$ denote the information set up to time k for $k=0,1,\ldots,T-1$. An investment strategy $u=\{u_k;k=0,1,\ldots,T-1\}$ is called an *admissible investment strategy* if u_k is finite and progressive measurable with respect to $\mathcal{F}_k$ for $k=0,1,\ldots,T-1$.

Denote $S_k=x_k-l_k$, $k=0,1,\ldots,T$, where S_k is called the surplus of the investor at time k. Assume that the investor's target is to find an optimal admissible investment strategy to minimize the risk of the terminal surplus $S_{T\wedge\tau}=x_{T\wedge\tau}-l_{T\wedge\tau}$ for a given level d of the expected terminal surplus, i.e., $E[S_{T\wedge\tau}]=d$, where the risk is measured by variance, i.e.,

$$Var[S_{T\wedge\tau}]=E\left[S_{T\wedge\tau}^2\right]-E^2[S_{T\wedge\tau}]=E\left[S_{T\wedge\tau}^2\right]-d^2.$$