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Dynamic optimal capital growth with risk constraints

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ABSTRACT

In this paper, risk metrics in capital growth and drawdown as a financial risk measure were considered. Moreover, we developed a dynamic portfolio management model with constraints on the maximal drawdown. Exact optimization algorithms run into difficulties in this framework and this motivates the investigation of simulated annealing optimized algorithm to solve the problem of maximizing long term growth of simultaneous risky investment. Empirical research indicates that the approach is inspiring for this class of portfolio optimization problems.

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1. Introduction

Optimal capital growth is a longstanding issue in both practical portfolio management and academic research on portfolio theory. Early work on this topic was mainly focused on the mean–variance approach (Markowitz, 1952). An alternative approach was proposed by Kelly (1956) who maximized the long term growth rate of the investor's capital. This was called Kelly capital growth theory or log strategy. Breiman (1961) proved that log strategy will, in the long run, beat any different strategy, almost surely. Subsequent works can be found in Latane (1959), Thorp (1969), Brown et al. (2006), and Browne and Whitt (1996).

Measures of risk have a crucial role in optimization under uncertainty, especially in coping with the losses that might be incurred in finance. Value at risk is a popular measure of risk which has been written into industry regulations. Other measures such as expected shortfall were suggested as practicable and alternative to value at risk. Feller (1950) investigated the ruin risk. Zhao et al. (2004) described risk metrics in capital growth process. In recent decades, general approaches to risk measure have been proposed (for details of this field see Uryasef et al., 2004).

The mean variance optimization is the cornerstone of modern finance theory, but it seems astonishing that investment practitioners do not put it to use more often. One argument often advanced is that expected returns, risks and correlations are measured with substantial error. Another criticism is the assumption of the normality of the returns, which is dismissed by practitioners in order to account for the fat tailedness and the asymmetry of the asset returns. Capital growth theory was applied by many institutional investors to run a superior hedge fund (Ziemba and Ziemba, 2007). However, the major shortcoming of the original capital growth approach is its lack of risk metric mechanism. From a practical point of view, the original model may often be considered too basic, which can not been applied to investment.

The present work is an extension of the basic model with risk constraints in capital growth process. We will show that a number of well known risk measures, including the risk of ruin measure, traditional variance measure, and shortfall measure, are special cases of the drawdown approach. The original model ignores many of the constraints faced by real world investors: trading limitations, size of the portfolio, risk measure, etc. It is more difficult to solve the optimal model with these constraints and analytical approach fails to solve the problem. Therefore, we investigate the simulated annealing to solve the problem of maximizing long term growth. The solution has widespread applications in asset allocation, which reduce the bias to a sufficiently small level.

This paper is organized as follows. In Section 2 we briefly overview the capital growth theory and related results. In Section 3 we investigate the risk metrics and drawdown in asset allocation. In Section 4 we develop a dynamic portfolio management model with drawdown constraints and investigate the optimized algorithm. Finally in Section 5, we finished empirical research in real financial data and output performance and risk results. We show that in consequence, this model and optimized algorithm are valuable to institutional investors.

2. Short summary of the capital growth theory

Let the initial bankroll be W_0. We assume that the investor has found a positive expectation game and is able to play this game repeatedly for n iterations, after which the bankroll is W_n. Suppose that our winning probability is p and the probability of losing is $q =$

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$1-p$. Game return R_i is defined as $R_i=(W_i-W_{i-1})/W_{i-1}$ where W_i is the wealth after i turns.

The investor bets fraction f_i of the actual wealth in each turn. After n turns the investor's wealth is equal to

$$W_n = W_0\prod_{i=1}^{n}(1+f_iR_i). \tag{1}$$

Because the returns are independent, the average wealth after n turns can be written as

$$\langle W_n\rangle = W_0\prod_{i=1}^{n}(1+E[f_iR_i]). \tag{2}$$

Since the game has a positive expectation, $E[f_iR_i]>0$ in this situation, in order to maximize $\langle W_n\rangle$ we would maximize $E[f_iR_i]$ at each trial. The optimal strategy is to stake everything in each turn. However, the probability of ruin is given by $1-p^n$, $\lim_{n\to\infty}(1-p^n)=1$ so ruin is almost sure. Thus maximization of $\langle W_n\rangle$ is not a good criterion. An asymptotically optimal strategy was first proposed by Kelly (1956).

Kelly chose to maximize the exponential growth rate of the investor's wealth

$$\lim_{n\to\infty}\frac{1}{n}\ln\frac{W_n}{W_0} = \lim_{n\to\infty}\frac{1}{n}\sum_{n=1}^{n}\ln(1+f_iR_i) = \langle\ln(1+f_iR_i)\rangle. \tag{3}$$

We refer to a trade's reward to risk ratio as an R multiple, which is simply a symbol for the initial risk. The Kelly strategy assumes that the probability of success and the pay off ratio are likely to vary across trades. The trader knows the likely reward and permissible risk on a trade before its initiation. Based on past experience, the trader can estimate the probability of success.

For the risky game introduced above, Eq. (3) can be rearranged as

$$G(f) = p\ln(1+af)+(1-p)\ln(1-bf) \tag{4}$$

where a and b are R multiples. Note that

$$G'(f) = \frac{pa}{1+af}-\frac{(1-p)b}{1-bf}=0 \tag{5}$$

when $f=f^*=\frac{pa+pb-b}{ab}$. Now

$$G''(f) = -\frac{pa^2}{(1+af)^2}-\frac{(1-p)b^2}{(1-bf)^2}<0. \tag{6}$$

$G(f)$ has a unique maximum at $f=f^*$, where $G(f^*)=p\ln(pa+pb)+(1-p)\ln\frac{(a+b)(1-p)}{a}$. Moreover, $G(f_c)=0$ so we get unique number $f_c>0$, where $0<f^*<f_c<1$.

After n turns the log of average bankroll will tend to $G(f^*)n$ times as much money as it started with. Numbers of trade equal to

$$n = \frac{1}{G(f^*)}\ln\frac{W_n}{W_0}. \tag{7}$$

Suppose the random variable X has a sample description space of m values, $x_1,\ldots,x_m$, with relevant probabilities $p_1,\ldots,p_m$. Population geometric mean return can be written as

$$\mathrm{PGMR} = \prod_{i=1}^{m}(x_i)^{p_i}-1. \tag{8}$$

In the case when the random variable is discrete it is easy to show that

$$\ln(1+\mathrm{PGMR}) = \sum_{i=1}^{m}p_i\ln x_i = E[\ln X]. \tag{9}$$

So, maximizing PGMR is equivalent to maximizing $E[\ln X]$. For any strategy

$$\mathrm{PGMR} = e^{E[\ln X]}-1. \tag{10}$$

Thorp (1969) shows that the investor's fortune will exceed any fixed bound when f is chosen in the internal $(0,f_c)$. But, if $f>f_c$, ruin is almost sure. If $f=f_c$, W_n will oscillate randomly between 0 and $+\infty$. Breiman (1961) shows that the strategy of maximizing $E[\log X_n]$ will, in the long run, beat any significantly different strategy, almost surely.

Samuelson (1971) states that $E[\log]$ maximization is for the special utility function $u(w)=\log w$ and not for other utility functions. Samuelson's criticisms are with the pure theory and not in conflict with any of the conclusions of the Kelly criterion. He makes a few points in his paper (Samuelson, 1979): first, those who follow the rule maximization of mean log of wealth with higher and higher probability will have more wealth in the long run than those who use an essentially different strategy. Second, some of those who have favorable asset returns period by period and maximize the expected log of wealth can lose a lot. Third, the expected log maximizing strategy is in some sense better than a strategy based on some other utility function.

Markowitz (1976) argues that when one traces out the set of mean variance efficient portfolios, which gives approximately the maximum $E[\log X]$. Young and Trent (1969) proved that $E[\log(X_i)]\approx \log E-\frac{1}{2}\left\{\frac{V}{E_2}\right\}$, where $X=$ gross return, $E=E[X]$ and $V=VarX$. Markowitz argues that the Kelly portfolio should be considered the upper limit for conservative choice among E,V efficient portfolios. An investor might prefer a lower mean and variance giving up return in the long run for stability in the short run.

Kelly has essentially zero risk aversion since its Arrow Pratt absolute risk aversion index is

$$-u''(w)/u'(w) = \frac{1}{w} \tag{11}$$

which is essentially zero. The absolute risk aversion is a measure of the curvature of an individual's utility function. Hence it never allocates more than the Kelly optimal fraction because then risk increases and growth decreases. As you allocate more and more above the Kelly optimal fraction, its properties become worse and worse. When you allocate exactly twice the Kelly optimal fraction, then the growth rate is zero plus the risk free rate. If you allocate more than double the Kelly criterion, then you will have a negative growth rate. Long term capital is an example of overbetting leading to disaster.

Thus you must either allocate Kelly or less. We call allocating less than Kelly fractional Kelly, which is simply a blend of Kelly and cash. Consider the negative power utility function $\delta\omega^\delta$ for $\delta<0$. This utility function is concave and when $\delta\to 0$ it converges to log utility. As δ gets larger negatively, the investor is less aggressive since his Arrow Pratt risk aversion is higher. For a given δ and $\alpha=\frac{1}{1-\delta}$ between 0 and 1, α is invested in the Kelly portfolio and $1-\alpha$ is invested in cash.

3. Risk metrics in capital growth process and drawdown

Measures of risk have a crucial role in optimization under uncertainty, especially in coping with the losses that might be incurred in asset management industry. In this section, a drawdown approach to investment risk was investigated. Drawdown risk measures are