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On mean-variance portfolio selection under a hidden Markovian regime-switching model

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ABSTRACT

We study a mean-variance portfolio selection problem under a hidden Markovian regime-switching Black–Scholes–Merton economy. Under this model, the appreciation rate of a risky share is modulated by a continuous-time, finite-state hidden Markov chain whose states represent different states of an economy. We consider the general situation where an economic agent cannot observe the "true" state of the underlying economy and wishes to minimize the variance of the terminal wealth for a fixed level of expected terminal wealth with access only to information about the price processes. By exploiting the separation principle, we discuss the mean-variance portfolio selection problem and the filtering-estimation problem separately. We determine an explicit solution to the mean-variance problem using the stochastic maximum principle so that we do not need the assumption of Markovian controls. We also provide robust estimates of the hidden state of the chain and develop a robust filter-based EM algorithm for online recursive estimates of the unknown parameters in the model. This simplifies the filtering-estimation problem.

1. Introduction

Portfolio selection is of importance in both the theory and practice of modern finance. Markowitz (1952) pioneered the development of quantitative models for portfolio selection and provided a mathematically elegant way to formulate the optimal portfolio selection problem. To simplify the issue and to convey the intuition of the problem, Markowitz considered a single-period economy and formulated the optimal portfolio selection as a static mean-variance optimization problem, where the variance, (or standard deviation), is used as a measure of risk. More specifically, the optimal portfolio selection becomes one of minimizing the variance of the portfolio's return for a fixed level of the expected portfolio's return, or one of maximizing the expected portfolio's return for a fixed level of the variance of the portfolio's return. The novelty of the Markowitz approach is that it simplifies the optimal portfolio selection problem to one where only two "statistics" matter, namely, the mean and the variance of the return from a portfolio. This simplified framework is justified when the distribution of returns is normal, or when an economic agent has a quadratic utility. The Markowitz mean-variance paradigm is also an ideal framework to convey the intuition of some important investment concepts, such as investment opportunities, portfolio diversification, efficient frontiers and portfolios. They can be represented either geometrically or analytically in a simple manner. These important concepts laid down the theoretical foundation for the Capital Asset Pricing Model developed independently by Treynor (1962), Sharpe (1964), Lintner (1965) and Mossin (1966). Despite its theoretical importance and its intuitive appeal, the Markowitz model is a single-period model, which makes it difficult to apply to the portfolio selection problem, since practical portfolio selection problems involve inter-temporal decisions.

The works of Merton (1969, 1971) pioneered the study of the optimal portfolio selection problem in a continuous-time framework. Merton first used stochastic optimal control theory, in particular, the dynamic programming approach, to derive a mathematically elegant and theoretically sound solution to optimal portfolio selection. The continuous-time framework of Merton provides more rigor and sharper results than discrete-time, multi-period models for portfolio selection. Under certain assumptions, such as a perfect market and a geometric Brownian motion (GBM) for asset price dynamics, Merton obtained a closed-form solution for the optimal portfolio selection problem. He showed, it is optimal to invest a constant proportion of wealth in a risky asset, where the constant is the well-known Merton ratio. This ratio can be used to justify some traditional portfolio decisions made in practice. For example, a rule of thumb in asset

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allocation of a pension fund is to allocate 60% in ordinary shares and 40% in fixed interest securities. This rule of thumb can be justified by the Merton ratio, (see, for example, Gerber and Shiu, 2000). A key assumption of the Merton model is the GBM assumption for the price dynamics of the risky share. However, many empirical studies reveal that the GBM assumption does not provide a realistic description of some "stylised" empirical features of assets' returns, such as the heavy-tailedness of the distribution of returns and time-varying conditional volatility. Under the GBM assumption, the investment opportunity set remains constant over time. However, more economic insights and implications can be gained if the investment opportunity set is allowed to vary stochastically over time. So, from both an economic and a statistical perspective, it is beneficial to develop a model which incorporates the empirical features of assets' returns and describes the stochastic evolution of investment opportunity sets.

In recent years, Markovian regime-switching models have become popular in economics, finance and actuarial science. Hamilton (1989) pioneered the econometric applications of Markovian regime-switching models. He introduced a class of discrete-time, Markov-switching autoregressive time series models, which can incorporate the structural changes of the hidden states of an underlying economy in the asset price dynamics. Many empirical studies reveal that Markovian regime-switching models provide a good fit for economic and financial time series. In particular, Markovian regime-switching models can provide reasonably good descriptions for some important empirical features of assets' returns. The class of Markovian regime-switching models also has the advantage that it can incorporate the stochastic evolution of investment opportunity sets over time. This is desirable from an economic perspective and some interesting and important economic insights and implications can be obtained from the Markovian regime-switching models. Indeed, Markovian regime-switching models have found diverse applications in finance. Some of these applications include Elliott and van der Hoek (1997) for asset allocation, Pliska (1997) and Elliott et al. (2001) for short rate models, Elliott and Hinz (2002) for portfolio analysis and chart analysis, Naik (1993), Guo (2001), Buffington and Elliott (2002a,b) and Elliott et al. (2005) for option valuation, Elliott et al. (2003) for volatility estimation, and others.

Applications of Markovian regime-switching models for portfolio selection have recently received some attention. Zhou and Yin (2003) and Yin and Zhou (2004) considered the Markowitz mean-variance portfolio selection problem under both discrete-time and continuous-time Markovian regime-switching models. They supposed that the underlying Markov chain that drives the Markovian regime-switching model is observable and formulated the problem as a linear-quadratic optimization problem. They obtained analytical results for efficient frontiers and interesting economic implications. Jang et al. (2007) studied the optimal consumption/investment problem for a Markovian regime-switching model driven by an observable underlying Markov chain when transaction costs are present. They obtained some novel and interesting insights into the effect of transaction costs on liquidity premia under some assumptions for the stochastic investment opportunity set described by the regime-switching. Elliott and Siu (in press) investigated a portfolio risk minimization problem under a Markovian regime-switching model. They formulated the problem as a stochastic differential game and provided a verification theorem for the solution of the Hamilton–Jacobi–Bellman, (HJB), equation. Some works on relaxing the assumption of an observable Markov chain have been given. For example, Sass and Haussmann (2004) considered a portfolio optimization problem in a multi-stock Markovian regime-switching market, where the drifts of the stocks are modulated by a continuous-time, finite-state hidden Markov chain. They provided an explicit representation of the optimal portfolio policy by exploiting the martingale approach for portfolio selection together with Malliavin calculus. They also derived stochastic differential equations governing the unnormalized filters of the unobservable drift processes and applied the EM algorithm to estimate the model parameters. Rieder and Bäuerle (2005) considered a portfolio optimization problem under a Markov-modulated geometric Brownian motion with an unobservable drift process modulated by a continuous-time, finite-state hidden Markov chain. They employed the HJB dynamical programming approach to solve the portfolio optimization problem and derived the stochastic differential equation governing the filter process of the unobservable drift process. The implementations of the filters and the estimation procedures in both Sass and Haussmann (2004) and Rieder and Bäuerle (2005) involve approximating stochastic integrals, which may not be easy. Comparing the optimal consumption/investment problem under Markovian regime-switching models with unobservable Markov chains, it seems that there is a relatively little work on the mean-variance portfolio selection problem under a hidden Markovian regime-switching model.

In this paper, we investigate a mean-variance portfolio selection problem under a hidden Markovian regime-switching Black–Scholes–Merton economy with two primitive securities, namely, a fixed interest security and an ordinary share. We suppose that the appreciation rate of the ordinary share is modulated by a continuous-time, finite-state hidden Markov chain whose states represent different states of an economy. Consequently, the appreciation rate of the risky share is unobservable to an economic agent. We consider the situation where the economic agent wishes to maximize the variance of the terminal wealth for a fixed level of the expected terminal wealth by allocating wealth to the fixed interest security and the ordinary share over time. By making use of the separation principle for filtering and control, the mean-variance portfolio selection problem and the filtering-estimation problem are solved separately. We adopt the stochastic maximum principle to derive an explicit solution to the mean-variance portfolio problem. This provides a direct derivation for the optimal portfolio policy. We derive filters for the hidden states of the economy. By exploiting the gauge transformation technique introduced by Clark (1978), the numerical work is simplified and a system of ordinary differential equations providing robust filters is obtained. We also develop the robust filter-based EM algorithm for online recursive estimates of the unknown parameters in the model. The implementations of the filters and the EM algorithm do not involve stochastic integrals, so they are more easy to implement.

This paper is structured as follows. In Section 2, we present the price dynamics of the primitive assets and describe some assumptions underlying the model. We also describe the mean-variance portfolio selection problem. Section 3 presents the use of the separation principle for control and filtering to separate the mean-variance portfolio selection problem and the filtering-estimation problem. In Section 4, we outline the derivations of the filters for the hidden states, and the robust EM algorithm to obtain the online recursive estimates of the unknown model parameters. Section 5 describes the stochastic maximum principle for solving the mean-variance portfolio selection problem.

2. Model dynamics and portfolio selection problem

In this section, we consider a continuous-time economy with two primitive securities, namely, a fixed interest security and an ordinary share. These primitive securities are tradeable continuously over time on a finite-time horizon $\mathcal{T} := [0, T]$, where $T < \infty$. We model uncertainty using a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where $\mathcal{P}$ is a base probability measure from which a family of absolutely continuous probability measures is generated.[1] We adopt bold-face letters to denote matrices, (or vectors), and write $\mathbf{y}'$ for the transpose of a matrix, or a vector $\mathbf{y}$.

[1] Here we use the term "base probability measure" instead of "reference probability measure" since "reference probability measure" has a special meaning in the filtering method we shall describe in the later part of this paper and we wish to reserve this term for describing the filtering technique.