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Asset allocation under stochastic interest rate with regime switching

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ABSTRACT

We investigate an optimal asset allocation problem in a Markovian regime-switching financial market with stochastic interest rate. The market has three investment opportunities, namely, a bank account, a share and a zero-coupon bond, where stochastic movements of the short rate and the share price are governed by a Markovian regime-switching Vasicek model and a Markovian regime-switching Geometric Brownian motion, respectively. We discuss the optimal asset allocation problem using the dynamic programming approach for stochastic optimal control and derive a regime-switching Hamilton–Jacobi–Bellman (HJB) equation. Particular attention is paid to the exponential utility case. Numerical and sensitivity analysis are provided for this case. The numerical results reveal that regime-switches described by a two-state Markov chain have significant impacts on the optimal investment strategies in the share and the bond. Furthermore, the market prices of risk in both the bond and share markets are crucial factors in determining the optimal investment strategies.

1. Introduction

Optimal asset allocation is of great importance in both the theory and practice of modern banking, finance and insurance. The pioneering work on optimal asset allocation was done by Markowitz (1952), where an elegant mathematical framework to an optimal asset allocation problem was established. To simplify the issue and to convey financial intuition of the problem, Markowitz considered a simple, single-period economy and formulated the optimal portfolio selection problem as a static mean–variance optimization problem. In such a problem, the variance, (or standard deviation), is used as a measure for risk. A portfolio that gives the maximum expected return for a fixed level of risk, or vice versa, is an optimal portfolio. In this sense, the Markowitz mean–variance paradigm simplifies the optimal asset allocation problem to one where only two “statistics” matter, namely, the mean (the expected return) and the variance (the risk) from a portfolio. It appears that the Markowitz mean–variance model is a statistical approach to investigate the optimal asset allocation. However, it has many far-reaching economic or financial implications for the optimal portfolio selection problem. Indeed, some fundamental concepts in modern finance theory including investment opportunity sets, portfolio diversification and efficient frontiers were introduced in the Markowitz paradigm. These important concepts laid down the theoretical foundation for the Capital Asset Pricing Model developed independently by Treynor (1962), Sharpe (1964), Lintner (1965) and Mossin (1966).

In practice, both institutional and individual investors may make their investment decisions dynamically over time. Consequently, it is questioned whether the static, one-period, Markowitz model is a suitable model for describing investment decisions in reality. Merton (1969, 1971) considered an expected utility approach to study the optimal portfolio selection problem in a continuous-time modeling framework. He pioneered the use of stochastic optimal control theory, in particular, the dynamic programming approach, to discuss the optimal portfolio selection problem. Under certain assumptions, such as a perfect market and geometric Brownian motion (GBM) for asset price dynamics, Merton obtained a closed-form solution for the optimal portfolio selection problem. He showed that it is optimal to invest a constant proportion of wealth in a risky asset, where the constant proportion is known as the Merton ratio. This ratio can be used to justify some traditional portfolio decisions made in practice. For example, a rule of thumb in asset allocation of a pension fund is to allocate 60% in ordinary shares and 40% in fixed interest securities. This rule of thumb may be justified by the Merton ratio, (see, for example, Gerber and Shiu (2000)).

Although Merton's approach to asset allocation problem produces some nice results, for example, a closed-form solution to the problem, there are some shortcomings, especially when we apply his approach to some institutional long-term investors, such as life insurers, pension funds and mutual funds. In the Merton model, the interest rate is assumed to be constant, which is untenable in recent years. Ever since the oil crisis in the last century, the interest rate has appeared more volatile in western countries. In addition, the marketization of the interest rate globally also contributes to more frequent and violent fluctuations in interest rates all over the world. Furthermore, with a view to diversifying risk, long-term investors usually allocate a certain proportion of their wealth in bonds besides

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bank deposits and shares as described in the Merton model. For a portfolio consisting of bonds, its duration, (i.e., the sensitivity of the value of the portfolio to the interest rate), is crucial. Consequently, it is not unreasonable to incorporate stochastic interest rate in an optimal asset allocation problem. Previous works on asset allocation under a stochastic interest rate environment include Lioui and Poncet (2001), Korn and Kraft (2001, 2004), Fleming and Pang (2004), Detemple and Rindisbacher (2005), Pang (2006), Liu (2007), Noh and Kim (2011). For a monograph on bond portfolio optimization with stochastic interest rate, please refer to Puhle (2008).

Another shortcoming is that the Merton model imposes a Geometric Brownian Motion (GBM) assumption for the price dynamics of the risky share, which is not consistent with many empirical studies, especially in a long time span in which structural changes in macroeconomic conditions can lead to dramatic transitions in market fundamentals. Under the GBM assumption, the investment opportunity set remains constant over time. However, in the long run, adjusting the investment opportunity set stochastically over time according to the current economic condition seems a wiser strategy for a rational investor. Consequently, from both an economic and statistical perspective, it pays us dividends to develop a model which can incorporate empirical features of assets' returns and describe stochastic evolution of investment opportunity sets. Regime-switching models provide a natural and convenient way to model the effect of changes in macroeconomic conditions on price series and economic series. Hamilton (1989) pioneered the econometric applications of Markovian regime-switching models. In such models, one set of model parameters is in force at a particular time depending on the state of an underlying Markov chain at that time. The states of the chain represent different states of an economy. A switch of one set of model parameters to another set is triggered by a transition in the Markov chain from one state to another. So they are able to incorporate structural changes in the model dynamics, which are attributed to changes in (macro)-economic conditions and business cycles. Empirically, regime-switching models provide a good fit to many economic and financial time series, especially when one wishes to fit long financial and investment series, say over thirty to forty years. From an economic perspective, the investment opportunity set varies stochastically over time in a regime-switching model. Some important economic insights into the optimal portfolio allocation problem can be obtained when one moves from a constant investment opportunity set to a stochastic investment opportunity set. Indeed applications of regime-switching models in optimal asset allocation have been investigated by a number of authors. Some examples include Zhou and Yin (2003), Bäuerle and Rieder (2004), Yin and Zhou (2004), Rieder and Bäuerle (2005), Elliott et al. (2010), Zhang et al. (2010) and Siu (2011).

In this paper, we investigate an optimal asset allocation problem under a stochastic interest rate model with regime-switching. We take into account two key sources of risk faced by an investor, namely, interest rate risk and risk due to transitions in macroeconomic conditions. There are three primitive securities in the market, namely, a bank account, an ordinary share, and a zero-coupon bond. The price dynamics of the ordinary share are modeled as a Markovian regime-switching Geometric Brownian motion. We adopt the Vasicek model to describe stochastic movements of the short rate. We further assume that all of the coefficients in the dynamics of the short rate are modulated by a continuous time, finite state, Markov chain. The states of the chain represent different states of the (macro)-economic condition. The chain is observable and its states are proxies of different levels of observable macro-economic indicators such as Gross Domestic Product, Retail Price Index, Sovereign Credit Ratings and others. We consider the case where the investor aims to maximize the expected utility of his/her terminal wealth. Using the dynamic programming principle, the optimal asset allocation problem can be transformed into solving a regime-switching Hamilton–Jacobi–Bellman (HJB) equation. This provides a direct way to derive an optimal asset allocation strategy. For the case of an investor with an exponential utility function, the regime-switching HJB equation is discussed. Since it is difficult, if not impossible, to obtain a closed form solution for the optimal asset allocation strategy, which involves an exponential-type nonlinear equation, we provide the numerical solutions for the optimal investment strategies and conduct a sensitivity analysis versus both coefficients of absolute risk aversion and market prices of risk in a two-state Markov chain case. Furthermore, we provide a sensitivity analysis of the terminal expected utility versus the market price of risk, the maturity of the bond and the coefficient of absolute risk aversion.

The rest of the paper is structured as follows. In Section 2, we first present the model dynamics in a Markovian regime-switching stochastic interest rate market. Then we consider the valuation of a zero-coupon bond, which is entered into our model as an investment opportunity. In Section 3, we discuss the optimal asset allocation problem and derive the regime-switching HJB equation. Section 4 discusses the solution to the regime-switching HJB equation when the investor has an exponential utility. The final section gives concluding remarks.

2. Model dynamics and valuation

We consider a continuous-time financial model with a time parameter set $\mathcal{T} := [0, T]$, where $T<\infty$. To describe uncertainty, we consider a complete, filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathcal{P})$ where $\mathcal{P}$ is the real world probability measure and $\mathbb{F} := \{\mathcal{F}(t)|t\in\mathcal{T}\}$ is the right-continuous, $\mathcal{P}$-completed, filtration. We shall give a more precise definition for the filtration $\mathbb{F}$ later in this section. Intuitively, the filtration $\mathbb{F}$ describes how uncertainty in the financial model resolves over time. It is well known that the market is incomplete under a stochastic interest rate environment. According to the second fundamental theorem of asset pricing, there is more than one equivalent martingale measure in an incomplete market. To select an equivalent martingale measure $\tilde{\mathcal{P}}$, we shall follow the developments of Korn and Kraft (2001) and Puhle (2008), where the market prices of risk of both the bond and share markets were assumed known. In the sequel, we shall first present the stochastic short rate dynamics and then discuss the concept of stochastic flows for deriving an exponential affine formula for the bond price.

2.1. The models

We model the evolution of the state of an economy over time by a continuous-time, finite-state, observable Markov chain $\mathbf{X} := \{\mathbf{X}(t)|t\in\mathcal{T}\}$ on $(\Omega, \mathcal{F}, \mathcal{P})$ taking values in the state space $S:=\{\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_N\}$, where $N\geq 2$ and $\mathcal{P}$ is the real-world probability measure. Without loss of generality, we adopt the formalism introduced by Elliott et al. (1994) and identify the state space of the chain by a set of standard unit vectors $\mathcal{E} := \{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_N\}\subset\mathcal{R}^N$. Here the j^{th} component of $\mathbf{e}_i$ is the Kronecker delta δ_{ij} for each $i, j=1, 2, \ldots, N$. We call the set $\mathcal{E}$ the canonical state space of the chain $\mathbf{X}$. Let $\mathbf{Q}:=[q_{ij}]_{i,\ j=1,\ 2,\ \ldots,\ N}$ be the generator of the chain $\mathbf{X}$ under the real-world probability measure $\mathcal{P}$. The generator $\mathbf{Q}$ is also called a rate matrix, or a Q-matrix. It describes statistical properties, or the probability laws, of the chain. For each $i, j=1, 2, \ldots, N$, q_{ji} is the constant, instantaneous, intensity of a transition of the chain $\mathbf{X}$ from state $\mathbf{e}_i$ to state $\mathbf{e}_j$. Note that $q_{ji}\geq 0$, for $i\neq j$, and that $\sum_{j=1}^{N} q_{ji}=0$, so $q_{ii}\leq 0$. Let $\mathbf{y}'$ be the transpose of a matrix, or a vector, $\mathbf{y}$. With the canonical state space $\mathcal{E}$ of the chain, Elliott et al. (1994) obtained the following semi-martingale dynamics for $\mathbf{X}$:

$$\mathbf{X}(t) = \mathbf{X}(0) + \int_0^t \mathbf{Q}'\mathbf{X}(u)du + \mathbf{M}(t).$$