Economic Modelling 26 (2009) 392–403

Contents lists available at ScienceDirect

Economic Modelling

journal homepage: www.elsevier.com/locate/econbase

Using the Haar wavelet transform in the semiparametric specification of time series ☆

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ARTICLE INFO

Article history:
Accepted 13 August 2008

JEL classification:
E32
C14

Keywords:
Wavelets
Haar Basis
Semiparametric Estimation
Time Series

ABSTRACT

Using theoretical arguments for nonparametric wavelet estimation, we devise regression-based semiparametric wavelet estimators to dissect linear from nonlinear effects in a time series. The wavelet estimators localize in both time and frequency so that distortion due to outliers is lessened. Our regression-based approach also lends itself to ease of replication, clarity, flexibility, timeliness and statistical validity. We demonstrate the efficacy of the approach via rolling regressions on time series of quarterly U.S. GDP growth rates, monthly Hong Kong/ U.S. exchange rates, weekly 1-month commercial interest rates and daily returns on the S&P 500.

1. Introduction

The evidence as to whether there are important nonlinear characteristics in economic time series is somewhat mixed. For example, Potter (1995) and Pesaran and Potter (1997) find evidence of important asymmetries in the responses of U.S. output to negative and positive shocks. On the flip side, Anderson and Vahid (2001) observe that formal tests generally fail to reject linearity in U.S. GDP, and point to the poor forecast performance of nonlinear models. Since nonlinear models may significantly overfit the data, the cost of using them may be high in terms of mean-absolute prediction error if the true process is linear.

In this paper we use the nonparametric Haar Wavelet Transform (HWT) to motivate a simple semiparametric estimator. Designed by A. Haar in 1910, the Haar basis was the first of the wavelet filters. Nonparametric estimation with the HWT is easily accomplished through regression analysis, and we extend the logic for the *non*-parametric case to *semi*parametric regression. As the Haar wavelet can be used to approximate any function in $L_2(\mathfrak{R})$, we argue below that the Haar father wavelet is useful in processing signals. Through thresholding, we can compress the number of coefficients needed to reproduce the important features of the signal.

We choose to use the HWT in this paper for a number of reasons. First, it is extraordinarily simple in concept and execution. As we shall see, however, simplicity does not necessarily imply poor performance. Since we use a semiparametric approach for which a traditional autoregressive process models the smooth transitions, the HWT is employed primarily to model non-smooth regime shifts. Although we could have employed more advanced wavelets, for our purposes the plain vanilla Haar basis is much easier to motivate, sufficiently flexible, and easy to directly employ in *semi*parametric regression. Finally, as with all wavelets, constructing the Haar basis follows from a well-defined procedure, thereby removing at least some of the investigator bias from the definitions of the step functions introduced below.

In effect, Haar scaling dummies are used to model intercept shifts. Based on an accumulation of evidence, Clements and Hendry (1999) assert that intercept corrections offer protection against structural breaks in macro-econometric models. Indeed, Clements and Hendry argue that such changes are the *dominant* cause of prediction failure in linear dynamic models, with the intercept shifts due to either changes in deterministic factors or changes in the mechanism for dynamic adjustment. The factors that induce the structural breaks may be institutional, political, financial or technological in nature.

The Haar scaling dummies are *ideally* suited for this modeling environment since they coincide with the usual regression variables employed to model intercept shifts. As per Härdle et al. (1998, Ch. 9), the differentiability of an optimal father wavelet estimator should match the assumed differentiability of the function being estimated. In our case, the HWT is very coarse and corresponds well to the intercept shifts that it approximates.

We make the assumption that our processes are stationary once the intercept shifts are recognized and modeled. In the case of cointegrated variables, that is, nonstationary variables that move in tandem, the HWT is applied to the error-correction representation. Likewise, to model changes in the slopes, the Haar scaling dummies interacts with

☆ I thank Vladimir Dobric and Richard Gundy for introducing me to the wavelet literature, and Michael Boldin, Mark Jensen, and Art King for providing helpful comments and insights.

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doi:10.1016/j.econmod.2008.08.001

selected regression variables. To wit, the semiparametric procedure in this paper can be modified to suit other modeling problems frequently encountered in practice. Too, the procedure can be implemented using standard statistical packages.

In this paper, *we will use italics to emphasize definitions and key points.* Section II describes a multiresolution analysis that employs the Haar wavelet coefficients, and Section III discusses thresholding the wavelet coefficients for the purpose of parsimony. In particular, we draw upon the results of Johnstone and Silverman (1997) to show that the thresholding technique is appropriate for correlated data. Section IV then extends the logic to semiparametric wavelet estimation.

The asymptotic distribution theory for our HWT estimator is analogous to that for a series estimator. It follows that Theorem 5.2 of Pagan and Ullah (1999) applies to the HWT estimator, along with the results from Andrews (1994) to justify its use for time series. Section 5 then uses simulation to establish the finite-sample forecasting ability of our proposed method. We employ quarterly U.S. GDP growth rates, monthly Hong Kong/U.S. exchange rates, weekly 1-month commercial interest rates and daily returns on the S&P 500 index. Section 6 concludes the paper.

2. The Haar wavelet transform

2.1. Wavelets defined

Our explanation of the Haar basis follows that of Aboufadel and Schlicker (1999), Härdle et al. (1998), and Thullard (2001). Define: $\int=\int_{-\infty}^{\infty}$, $\sum_i=\sum_{i=0}^{\infty}$, $Z=\{\ldots,-1,0,1,\ldots\}$, and let $L_2(\Re)$ be the space of all real valued functions, f, on $\Re$ such that the L_2-norm for f is finite:

$$\|f\|_2=\left(\int|f(x)|^2dx\right)^{1/2}<\infty \tag{1}$$

A function ψ is called a wavelet if there exists some function ψ^c such that any function $f\in L_2(\Re)$ can be written as,

$$f(x)=\sum_{m,n}<f,\psi^c_{m,n}>\psi_{m,n} \tag{2}$$

where $<f, g>$ represents the scalar product,

$$<f,g>=\int f(x)g(x)dx \tag{3}$$

A function $\psi\in L_2(\Re)$ is called an *orthogonal wavelet* and the associated system of functions $\{\psi_{m,n},\ m,n\in Z\}$an *orthonormals basis* of $L_2(\Re)$, if first,

$$<\psi_{a,b}\,\psi_{c,d}>=\delta_{a,c}\cdot\delta_{b,d},\ \text{where } \delta_{j,k} \text{ is the Kronecker delta,} \tag{4}$$

second, if every $f\in L_2(\Re)$ can be written as

$$f(x)=\sum_{m,n}<f,\psi_{m,n}>\psi_{m,n} \tag{5}$$

and finally, if

$$\psi_{m,n}(x)=2^{m/2}\cdot\psi(2^m\cdot x-n) \tag{6}$$

We define the wavelet coefficients as $c_{m,n}\equiv<f,\psi_{m,n}>$ for Eq. (5) and require that $\sum_{m,n}|c_{m,n}|^2<\infty$. Note that for an orthogonal wavelet, $\psi^c=\psi$. The simplest orthogonal wavelet is the Haar wavelet, defined as,

$$\psi(x)=\begin{cases}1, & x\in\left(0,\frac{1}{2}\right]\\ -1, & x\in\left(\frac{1}{2},1\right]\\ 0, & \text{otherwise}\end{cases} \tag{7}$$

An associated wavelet is the Haar *scaling function*, defined as,

$$\phi(x)=I(x\in(0,1]) \tag{8}$$

where $I(A)=1$ if A is satisfied, and zero otherwise. With

$$\phi_{m,n}(x)=2^{m/2}\cdot\phi(2^m\cdot x-n) \tag{9}$$

we can alternatively approximate any function in the space $L_2(\Re)$ as a linear span of the system of functions $\{\{\phi_{0,n}\},\{\phi_{1,n}\},\ldots\}$. For a proof, see Proposition 2.1 in Härdle et al. (1998). We sometimes refer to $\phi(x)$ as the Haar *father* wavelet and $\psi(x)$ as the Haar *mother* wavelet. We will explain the close relationship between the father and mother wavelets later in the paper. For now we concentrate on the father wavelet — a function useful in approximating functions in $L_2(\Re)$ that can also be viewed as a step function in a regression equation.

Consider, for example, that $\phi_{0,n}(x)=\phi(x-n)$ spans the subspace of $L_2(\Re)$ defined as

$$V_0=\{f\in L_2(\Re): f \text{ is constant on } (n,n+1], n\in Z\} \tag{10}$$

That is, we can write any function in V_0 as

$$f(x)=\sum_n c_n\phi(x-n)=\sum_n c_n I(x-n\in(0,1]) \tag{11}$$

with the coefficients c_n given by

$$c_n=<f,\phi_{0,n}>=\int f(x)I(x-n\in(0,1])dx \tag{12}$$

Likewise, $\phi_{1,n}(x)=\sqrt{2}\,\phi(2x-n)$ spans the subspace of $L_2(\Re)$ defined as

$$V_1=\{f\in L_2(\Re): f \text{ is contant on } (n/2,(n+1)/2), n\in Z\} \tag{13}$$

We can write,

$$V_1=\{h(x)=f(2x): f\in V_0\} \tag{14}$$

In fact, $\{\phi_{0,n}\}$ forms an orthonormal basis for V_0 and $\{\phi_{1,n}\}$ forms an orthonormal basis for V_1. Too, $V_0\subset V_1$, in that V_0 is a linear subspace of V_1 since V_1 simply considers piecewise constants on a finer grid than V_0. In general, $\phi_{m,n}(x)=2^{m/2}\,\phi(2^m x-n)$ forms an orthonormal basis for

$$V_m=\{h(x)=f(2^m x): f\in V_0\} \tag{15}$$

with

$$V_0\subset V_1\subset\ldots\subset\ldots V_m\subset\ldots \tag{16}$$

By choosing a sufficiently large value of m, any function $f\in L_2(\Re)$ can be approximated as closely as we like by a piecewise constant function with constant values over the interval $(n/2^m,(n+1)/2^m]$. The approximation error is measured by the $L_2(\Re)$ norm:

$$d(f,g)=\|f-g\|=\left(\int[(f(t)-g(t)]^2dt\right)^{1/2} \tag{17}$$

where for our immediate purposes $f(x)$ is approximated by $g(x)$, the wavelet expansion. Choosing a small value of m reveals only the most salient features of $f(x)$, while choosing a large value of m will also reveal the details. We say that the sequence of spaces $\{V_m,\ m\in Z)$ generated by the Haar orthonormal system of scaling functions is a *multiresolution analysis* because $V_{m-1}\subset V_m$ and $U_{m\geq 0}\,V_m$ is dense in $L_2(\Re)$.

2.2. Signal processing

In economics we are frequently interested in observing some phenomenon measured at regular intervals over time. Examples of macroeconomic time series are national unemployment, the current account deficit, aggregate inventory levels and gross domestic product. The process of gathering data at these regular intervals allows us to form a string of numbers called a *signal*. The signal can