Economic Modelling 28 (2011) 1195–1201

Contents lists available at ScienceDirect

Economic Modelling

journal homepage: www.elsevier.com/locate/ecmod

Convergence of numerical solutions for a class of stochastic age-dependent capital system with Markovian switching

Zhang Qi-min

School Mathematics and Computer Science, Ningxia University, Yinchuan, 750021, People's Republic of China

ARTICLE INFO

Article history:
Accepted 7 December 2010

Subject classification (AMS):
60H10
60H05
60H35
65C30

Keywords:
Stochastic age-dependent capital system
Markovian switching
Numerical solution
Euler approximation

ABSTRACT

Recently, numerical solutions of stochastic differential equations have received a great deal of attention. Numerical approximation schemes are invaluable tools for exploring its properties. In this paper, we introduce a class of stochastic age-dependent (vintage) capital system with Markovian switching, and investigate the convergence of numerical approximation. It is proved that the numerical approximation solutions converge to the analytic solutions of the equations under the given conditions. A numerical example is provided to illustrate the theoretical results.

1. Introduction

Recently, the vintage capital model has become increasingly popular among economists, especially because it provides an appealing framework for the analysis of investment volatility. Some pioneering works on the deterministic models of age-dependent (vintage) capital have been reported, for instance (Hritonenko and Yatsenko, 2005; Feichtinger et al., 2006; Renan-Ulrich and Natali, 2008). By analyzing these deterministic models of age-dependent (vintage) capital, we are able to find out that economy growth model focuses on four variables: output, capital, labor, and technological progress. Capital, labor, and technological progress are combined to produce output. However, some important sources of uncertainty may be discontinuous, recurrent, and fluctuating. Such significant events include innovations in technique, introduction of new products, natural disasters, and changes in laws or government policies. When capital, labor, and technological progress in corporations also bring abrupt changes in their structure, the Markovian jump system is very appropriate to describe their dynamics (Wang et al., 2006; Li et al., 2009).

On the other hand, systems with Marvokian jumps have been attracting increasing research attention (Hu et al., 2006; Shi et al., 2005; Wang et al., 2006). This class of systems is a special class of hybrid systems, which is specified by two components in the state. The first one refers to the mode, which is described by a continuous-time finite-state Markovian process, and the second one refers to the state which is represented by a system of differential equations. The Markovian jump systems (MJSs) have the advantage of modeling the dynamic systems subject to abrupt variation in their structures, such as component failures or repairs, sudden environmental disturbance, changing subsystem interconnections, and operating in different points of a nonlinear plant (Mariton, 1990). In addition, there exists an extensive literature dealing with stochastic differential equations with discontinuous paths incurred by Levy processes (for instance, see monographs (Applebaum, 2004; Protter, 2004) and references therein). These equations are used as models in the study of queues, insurance risks, dams, and more recently in mathematical finance. In recent years, Markovian-switching models have been attracted much attention by researchers and practitioners in economics and finance (Elliott et al., 2001, 2007; Elliott and Hinz, 2002). These models are able to incorporate the structural changes in the model dynamics, which might be attributed to the changes in macroeconomic conditions and different stages of business cycles. Now, applications of Markovian switching models can be found in various important fields in financial economics. Some of these applications include Elliott et al. (2007) for asset allocation, Elliott et al. (2001) for short rate models, Elliott and Hinz (2002) for portfolio analysis and chart analysis, Guo (2001) and Buffington and Elliott (2002) for option valuation, Elliott et al. (2007) for pricing and hedging volatility and variance swapping, and others.

The spotlight has turned to the application of Markovian switching models to value options. Markovian switching models provide a more realistic way to describe the asset price dynamics for option valuation. They can incorporate the effect of structural changes in macro-economic conditions and business cycles on option valuation. In particular, the

E-mail address: zhangqimin64@sina.com.

0264-9993/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.econmod.2010.12.003

analytical pricing formula is given by the integral of the Black–Scholes–Merton formula and the occupation time of the regime-switching process. Guo (2001) introduced a novel option pricing approach under a Markovian switching geometric Brownian motion (GBM).

Due to the nonlinear feature of stochastic modeling with Markovian switching, and presently it is hard to bring out explicit solutions, to construct the efficient computational methods becomes extremely important. For example, Chenggui and Xuerong (2004) gave the convergence of the Euler–Maruyama method for stochastic differential equations with Markovian switching, Ronghua and Yingmin (2006) discussed the convergence of numerical solution to stochastic delay differential equation with Markovian switching, Shaobo and Fuke (2009) investigated the convergence of numerical solutions to neutral stochastic delay differential equations with Markovian switching under the local Lipschitz condition. The equations they considered are stochastic (delay) differential equations with Markovian switching.

However, to the best of our knowledge, there are not any numerical methods available for stochastic partial differential equations with Markovian switching. In this paper, we develop a class of stochastic age-dependent (vintage) capital system with Markovian switching, and use the recent mathematical technique on the stochastic population system to estimate its numerical solutions. We shall extend the idea from the papers (Shaobo and Fuke, 2009; Xuerong and Sotirios, 2003) to the numerical solutions for stochastic age-dependent capital system with Markovian switching. The main purpose of this paper is to investigate the convergence of numerical approximation of stochastic age-dependent capital system with Markovian switching under the given conditions. In Section 2, the stochastic age-dependent (vintage) capital system with Markovian switching is developed. In Section 3, we shall collect some basic preliminary results which are essential for our development and the Euler approximation analysis, and the Euler approximation is introduced. In Section 4, we give several lemmas which are useful for our main results. In Section 5, we shall show the main results that the numerical solutions will converge to the true solutions to stochastic age-dependent capital equations with Markovian switching under the given conditions. In Section 6, a numerical example is provided to illustrate the theoretical results. Conclusion is given in Section 7.

2. The model

The following deterministic models of age-dependent (vintage) capital may be described by (see Feichtinger et al., 2006; Renan-Ulrich and Natali, 2008):

$$\begin{cases} \dfrac{\partial K(a,t)}{\partial t}+\dfrac{\partial K(a,t)}{\partial a} & \\ \quad = -\mu(a,t)K(a,t)+f(t,K(a,t)). & \text{in } Q, \\ K(0,t)=\phi(t)=\gamma(t)A(t)F\left(L(t),\int_0^A K(a,t)da\right), & \text{in } t\in[0,T], \\ K(a,0)=K_0(a), & \text{in } a\in(0,A), \\ N(t)=\int_0^A K(a,t)da, & \text{in } t\in[0.T], \end{cases} \tag{1}$$

where $Q=[0;A]\times[0;T]$; the stock of capital goods of age a at time t is denoted by $K(a,t)$, $N(t)$ is the total sum of the capital, a is the age of the capital, the investment $\phi(t)$ in the new capital, and the investment $f(t,K(a,t))$ in the capital of age a are the endogenous (unknown) variables. The maximum physical lifetime of capital A, the planning interval of calendar time $[0,T)$, the depreciation rate $\mu(a,t)$ of capital, and the capital density $K_0(a)$ (the initial distribution of capital over age) are given. $\gamma(t)$ denotes the accumulative rate at the moment of t; $0<\gamma(t)<1$, and $A(t)$ is the technical progress at the moment of t. The stock of capital goods of age a at time t is denoted by $K(a,t)$. This makes that total output produced in year t defined as $N(t)=\int_0^A K(a,t)da$. In each sector all the firms have an identical neoclassical technology and produce output using labor and capital. The production function $F(L(t),\int_0^A K(a,t)da)$ is neoclassical, where $\int_0^A K(a,t)da$ is the total sum of the capital and $L(t)$ is the labor force. Eq. (1) is a generalization of the deterministic capital equation. Eq. (1) describes the evolution of the composition of the productive capital as a function of purchasing/selling new or used capital. According to Eq. (1), machines of any age between 0 and A can be bought or sold.

Evolutionary growth models are nondeterministic models, representing the uncertainty inherent to evolution in general and to innovation (diffusion) processes in particular. Usually, the models include a certain degree of uncertainty by representing the innovation process as a stochastic process. In the Nelson and Winter model (1982), the probability of techniques to be discovered is represented as a Markovian process. In Silverberg and Lehnert (1993), where innovation is defined in terms of newly entering firms, innovations are generated as random drawings from a normal probability distribution. In addition, actual access to a machine and imitation of machines of other firms occur in similar two stage stochastic processes. The stochastic age-dependent population system has been established by Qimin et al. (2004).

In order to describe this situation capital process, suppose that the parameter $f(t,K)$ is stochastically perturbed, with

$$f(t,K(a,t))=f(r(t),K(a,t))+g(r(t),K(a,t))\frac{dW_t}{dt}.$$

Such significant events include innovations in technique, introduction of new products, natural disasters, and changes in laws or government policies. Then the stochastic capital system may be described by

$$\begin{cases} \dfrac{\partial K(a,t)}{\partial t}+\dfrac{\partial K(a,t)}{\partial a} & \\ \quad = -\mu(a,t)K(a,t)+f(r(t),K(a,t))+g(r(t),K(a,t))\dfrac{dW_t}{dt} & \text{in } Q, \\ K(0,t)=\phi(t)=\gamma(t)A(t)F\left(L(t),\int_0^A K(a,t)da\right), & \text{in } t\in[0,T], \\ K(a,0)=K_0(a), & \text{in } a\in(0,A), \\ N(t)=\int_0^A K(a,t)da, & \text{in } t\in[0.T], \\ r(0)=i_0, & \end{cases} \tag{2}$$

uncertainty in the financial market is assumed to enter through the components of a Brownian motion W_t, and the components of a Markovian process. $r(t)$ is a Markovian chain taking values in $S=\{1,2,\cdots,N\}$.

A new stochastic differential equation model (2) for an age-dependent (vintage) capital system is derived. It is an extension of Eq. (1).

3. Preliminaries and approximation

Throughout this paper, we shall denote by $L^2([0,A])$ the space of functions that are square-integrable over the domain $[0,A]$. Let

$$V=H^1([0,A])\equiv\left\{\varphi|\varphi\in L^2([0,A]),\frac{\partial\varphi}{\partial a}\in L^2([0,A]),\right.$$
$$\left.\text{where }\frac{\partial\varphi}{\partial a}\text{ are generalized partial derivatives}\right\}.$$