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Uniqueness of equilibrium in two-player asymmetric Tullock contests with intermediate discriminatory power[☆]

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HIGHLIGHTS

- We study uniqueness of equilibrium in two-player Tullock contests.
- We study the case where the discriminatory power is in between 1 and 2.
- We provide a different approach to establish the uniqueness.
- The study fills up a remaining gap in equilibrium analysis in contests.

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ABSTRACT

This paper provides a different approach to establish the uniqueness of equilibrium in Tullock contests between two players with asymmetric valuations, when the discriminatory power r is between 1 and 2. Our result complements that of Ewerhart (2017) in filling up the remaining gap in the literature on the uniqueness of equilibrium in two-player asymmetric Tullock contests.

1. Introduction

Many competitions and social conflicts such as lobbying, school admission, war, sports and political campaigns can be modeled as contests.[1] Starting from Tullock (1980)'s seminal contribution, a classical family of contest models called Tullock contests has been widely adopted in the literature. In a Tullock contest, there is a discriminatory power $r \in (0, +\infty)$ which measures the effectiveness of players' efforts in determining their winning chances. In particular, two-player Tullock contests with diverse discriminatory powers are broadly applied in a variety of contexts, e.g., Klumpp and Polborn (2006), Wang (2010), Epstein et al. (2013), Franke and Leininger (2014), Fu et al. (2015, 2016), Zhang and Zhou (2016), Serena (2016) and Feng and Lu (2016) among many others.

One fundamental issue regarding Tullock contests and the applications is the existence and uniqueness of equilibrium. This issue has been addressed extensively in the literature. Nti (1999, 2004) considers the family of pure strategy equilibria and establishes that a unique pure strategy equilibrium exists if r is lower than a threshold $\hat{r} \in (1, 2]$, which is solely determined by the ratio of the two players' valuations (symmetric or asymmetric).[2] For $r \geq 2$, Baye et al. (1994) establish the existence of symmetric mixed

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1 See Konrad (2009) for a comprehensive review of the contest literature.

2 Pérez-Castrillo and Verdier (1992), Szidarovszky and Okuguchi (1997) and Cornes and Hartley (2005) established sufficient conditions for a unique pure-strategy equilibrium while allowing N symmetric or asymmetric players.

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strategy equilibrium, and Alcalde and Dahm (2010) show that there exists an all-pay equilibrium while allowing N asymmetric players, in which only the top two strongest players are active. Nevertheless, a closed-form solution of equilibrium is yet to be identified. Ewerhart (2015, 2017) further demonstrates that any equilibrium in a two-player Tullock contest with $r > 2$ yields the same equilibrium effort, winning probabilities, participation probabilities, and equilibrium payoffs. When r falls between $\hat{r}$ and 2, Wang (2010) identifies a semi-pure strategy equilibrium. Although this equilibrium has been utilized in a number of studies including Fu et al. (2016) and Feng and Lu (2016), its uniqueness has yet to be established. Same problem arises for Nti (1999, 2004)'s equilibrium when r is between 1 and $\hat{r}$, that is, it remains unknown whether there are equilibria that lie outside this class of pure strategy equilibria.[3] In this paper, we fill up this gap by providing the uniqueness result when $r \in (1, 2)$. In other words, we establish that Nti (1999, 2004)'s equilibrium is unique for $r \in (1, \hat{r}(\frac{v_j}{v_i})]$, and Wang (2010)'s equilibrium is unique for $r \in (\hat{r}(\frac{v_j}{v_i}), 2)$.

Our paper is closely related to Ewerhart (2017) who provides a first and elegant proof for the uniqueness of equilibrium when $r \in (1, 2]$.[4] While Ewerhart (2017)'s proof relies on a key property of interchangeability across equilibria of two-player contests, our proof allows us to see the direct implications of a mixed strategy equilibrium on players' payoffs, the first order conditions and second order conditions.

2. The analysis

We consider a standard two-player Tullock contest with asymmetric values and an arbitrary discriminatory power $r \in (0, +\infty)$. The two players are indexed respectively by i and j. Player i's value is v_i and player j's value is v_j. Without loss of generality, we assume $v_i > v_j > 0$.[5] We thus call player i the stronger player and player j the weaker player. Player i's winning probability is given by $p_i(x_i, x_j) = \frac{x_i^r}{x_i^r + x_j^r}$, where x_i and x_j denote players' effort. When $x_i = x_j = 0$, we assume $p_i = p_j = \frac{1}{2}$. We let $\mu_i(v_i, v_j; r)$ and $\mu_j(v_i, v_j; r)$ denote the players' equilibrium strategy.

Following Nti (1999, 2004), we define the following threshold of discriminatory power.

Definition 1. $\forall w \in (0, 1)$, a threshold $\hat{r}(w) \in (1, 2)$ is defined as the unique solution of equation $z = 1 + w^z$.

Given $v_i > v_j > 0$, Nti (1999, 2004) provides a unique pure strategy equilibrium for $r \leq \hat{r}(\frac{v_j}{v_i})$; Wang (2010) explicitly identifies a semi-pure strategy equilibrium for $r \in [\hat{r}(\frac{v_j}{v_i}), 2)$; and for $r \geq 2$, Alcalde and Dahm (2010) show the existence of an all-pay equilibrium and Ewerhart (2015, 2017) demonstrates its uniqueness in terms of equilibrium efforts, winning chances and players' payoffs. In this paper, we consider the case of $r \in (1, 2)$, and show that the equilibrium identified in the literature is unique without making any constraint on a player's strategy space. More specifically, we establish that the equilibrium identified by Nti (1999) is unique when $r \in (1, \hat{r}(\frac{v_j}{v_i})]$, and the equilibrium identified by Wang (2010) is unique when $r \in (\hat{r}(\frac{v_j}{v_i}), 2)$.

Theorem 1. $\forall r \in (1, 2)$*, the equilibrium is unique.*

Proof. Suppose that (μ_i, μ_j) constitutes an equilibrium (in mixed or pure strategy) for $r \in (1, 2)$. Clearly, we have $supp(\mu_i)$, $supp(\mu_j) \subseteq [0, v_j]$. We let $F_{\mu_i}(x) = \Pr(x_i \leq x|\mu_i)$ and $F_{\mu_j}(x) = \Pr(x_j \leq x|\mu_j)$. Let $\bar{x}_i = \sup_{x_i \in supp(\mu_i)}\{x_i\}$ and $\bar{x}_j = \sup_{x_j \in supp(\mu_j)}\{x_j\}$.

Our proof will proceed in three steps.

Step 1: We first eliminate the possibility that there exist at least two different $x_{i1}\ (> 0), x_{i2}\ (> 0) \in supp(\mu_i)$, but only one $x_j\ (> 0) \in supp(\mu_j)$. Without loss of generality, we assume that $0 < x_{i1} < x_{i2}$. Player i's expected payoff $\pi_i(x, \mu_j) = \frac{x^r}{x^r + x_j^r}(1 - F_{\mu_j}(0))v_i + F_{\mu_j}(0)v_i - x, \forall x > 0$. Both $x_{i1} (> 0)$ and $x_{i2}\ (> 0) \in supp(\mu_i)$ should satisfy the first order condition:

$$\frac{rx^{r-1}x_j^r}{(x^r + x_j^r)^2}(1 - F_{\mu_j}(0))v_i - 1 = 0, x = x_{i1}, x_{i2}.$$

However, there exists only one positive solution for the above equation, which is a local maximum.[6] Therefore, we can eliminate this possibility. Analogously, we can eliminate the possibility that there exist at least two different $x_{j1}\ (> 0), x_{j2}\ (> 0) \in supp(\mu_j)$, but only one $x_i\ (> 0) \in supp(\mu_i)$.

Step 2: We next eliminate the possibility that there exist at least two different $x_{i1}\ (> 0)$, $x_{i2}\ (> 0) \in supp(\mu_i)$, and at least two different $x_{j1}\ (> 0), x_{j2}\ (> 0) \in supp(\mu_j)$. Without loss of generality, we assume that $0 < x_{i1} < x_{i2}$ and $0 < x_{j1} < x_{j2}$.

$\forall x > 0$, player i's expected payoff $\pi_i(x, \mu_j)$ is:

$$\begin{aligned}\pi_i(x, \mu_j) &= \int_{[0, \bar{x}_j]} \frac{x^r}{x^r + x_j^r} v_i dF_{\mu_j}(x_j) - x \\ &= F_{\mu_j}(0)v_i + \int_{(0, \bar{x}_j]} \frac{x^r}{x^r + x_j^r} v_i dF_{\mu_j}(x_j) - x.\end{aligned}$$

Since x_{i1} and x_{i2} must render the same expected payoff to player i, i.e. $\pi_i(x_{i1}, \mu_j) = \pi_i(x_{i2}, \mu_j)$,[7] we have

$$x_{i1} - x_{i2} = \int_{(0, \bar{x}_j]} \frac{x_{i1}^r}{x_{i1}^r + x_j^r} v_i dF_{\mu_j}(x_j) - \int_{(0, \bar{x}_j]} \frac{x_{i2}^r}{x_{i2}^r + x_j^r} v_i dF_{\mu_j}(x_j). \quad (1)$$

Moreover, we prove that both $x_{i1}\ (> 0), x_{i2}\ (> 0) \in supp(\mu_i)$ must satisfy first order conditions:

$$x = \int_{(0, \bar{x}_j]} \frac{rx^r x_j^r}{(x^r + x_j^r)^2} v_i dF_{\mu_j}(x_j), x = x_{i1}, x_{i2}.$$

To show that, by definition of differentiation, $\forall x > 0$, we consider

$$\begin{aligned}&\lim_{h \to 0} \frac{1}{h}\left[\int_{(0, \bar{x}_j]} \frac{(x+h)^r}{(x+h)^r + x_j^r} v_i dF_{\mu_j}(x_j) - \int_{(0, \bar{x}_j]} \frac{x^r}{x^r + x_j^r} v_i dF_{\mu_j}(x_j)\right] \\ &= \lim_{h \to 0} \frac{1}{h}\left[\int_{(0, \bar{x}_j]} \left(\frac{(x+h)^r}{(x+h)^r + x_j^r} - \frac{x^r}{x^r + x_j^r}\right) v_i dF_{\mu_j}(x_j)\right] \\ &= \lim_{h \to 0} \frac{1}{h}\left\{\int_{(0, \bar{x}_j]} \frac{(x+h)^r[x^r + x_j^r] - x^r[(x+h)^r + x_j^r]}{[(x+h)^r + x_j^r][x^r + x_j^r]} v_i dF_{\mu_j}(x_j)\right\} \\ &= \lim_{h \to 0} \frac{1}{h}\left\{\int_{(0, \bar{x}_j]} \frac{[(x+h)^r - x^r]x_j^r}{[(x+h)^r + x_j^r][x^r + x_j^r]} v_i dF_{\mu_j}(x_j)\right\} \\ &= \lim_{h \to 0} \frac{1}{h}\left\{\int_{(0, \bar{x}_j]} \frac{[(x+h)^r - x^r]x_j^r}{[(x+h)^r + x_j^r][x^r + x_j^r]} v_i dF_{\mu_j}(x_j)\right\} \\ &= \lim_{h \to 0} \frac{1}{h}\left\{\int_{(0, \bar{x}_j]} \frac{[r(x+h)^{r-1}h + O(h^2)]x_j^r}{[(x+h)^r + x_j^r][x^r + x_j^r]} v_i dF_{\mu_j}(x_j)\right\}\end{aligned}$$

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3 When $r \leq 1$, as players' payoff functions are globally concave in their own effort given the other's (mixed) strategy, the equilibrium uniqueness follows immediately.

4 To our best knowledge, Ewerhart (2017) was first posted online on March 2, 2017. The methods of the two studies are different, and the two contributions are independent.

5 Otherwise, we could relabel the two players. We ignore the trivial case where the two players are symmetric.

6 Since $\frac{d}{dx}[\frac{rx^{r-1}x_j^r}{(x^r+x_j^r)^2}] = \frac{r}{(x^r+x_j^r)^3}x^{r-2}x_j^r[(-r-1)x^r+(r-1)x_j^r]$, which first decreases and then increases with x, there is one local minimum and one local maximum.

7 Please refer to Lemma A.8(ii) in Ewerhart (2017), which shows that both must equal exactly the equilibrium payoff. We thank Christian Ewerhart for directing us to this useful result.