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Determinacy analysis in high order dynamic systems: The case of nominal rigidities and limited asset market participation

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A B S T R A C T

We show how to use Hurwitz polynomials to study the stability and uniqueness of Rational Expectation equilibria (REE) in Dynamic General Equilibrium models (DGE). We apply this method to a model characterized by sticky wages and prices and by limited asset market participation (LAMP). We prove analytically in a fourth-order dynamics system that, once nominal wage stickiness is taken into account, LAMP does not invalidate the Taylor Principle.

1. Introduction

We show how to use Hurwitz polynomials to study the stability and uniqueness of REE in DGE models. We apply this methodology to a New Keynesian (NK) model featuring sticky wages, prices and LAMP. Under LAMP a portion of agents spend their labor income in each period. Bilbiie (2008) studies determinacy properties of interest rate rules in a NK economy with LAMP and a frictionless labor market. He shows that determinacy of the REE requires an inversion of the Taylor principle, that is, the nominal interest rate needs to react less than one-to-one to inflation. We assess analytically how nominal wage stickiness affects this result. We represent our model as a fourth order dynamic system. Despite this high order, our methodology delivers analytical conditions for the determinacy of the REE. Colciago (2011) shows numerically that wage stickiness helps restoring the Taylor Principle as a necessary condition for determinacy in the presence of LAMP. We prove analytically the generality of this numerical result.

2. Methodology

Consider a system of linear difference equations in the form

$$E_t z_{t+1} = A z_t, \tag{1}$$

where z_t is a $nx1$ vector including n_1 predetermined variables and n_2 non-predetermined variables, where $n = n_1 + n_2$. A is a nxn coefficient matrix with characteristic polynomial

$$P_C(\gamma) = \gamma^n + a_1\gamma^{n-1} + \cdots a_i\gamma^{n-i} + \cdots + a_{n-1}\gamma + a_n. \tag{2}$$

The stability and uniqueness of the solution to (1) depend on the location of the roots of $P_C(\gamma)$ inside the unit circle $|\gamma| < 1$. Blanchard and Kahn (BK henceforth, 1980) show that (1) has a stable and unique solution if (2) has n_2 roots larger than one in absolute value and n_1 roots lower than one in absolute value. Verifying if BK conditions are satisfied can be cumbersome, particularly so as n gets larger and if z_t contains both predetermined and non-predetermined variables.

Following Felippa and Park (2005) we transform the polynomial $P_C(\gamma)$ in an Hurwitz polynomial, $P_H(s)$, by applying the conformal involuntary transformation

$$\gamma = \frac{1+s}{1-s}. \tag{3}$$

Given (3), it is easy to check that $|\gamma| \leqslant 1 \Leftrightarrow s \leqslant 0$. Expanding the polynomial, one obtains a quotient of two polynomials:

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$\tilde{P}_H(s) = \frac{P_H(s)}{Q_H(s)}$ where the roots of $\tilde{P}_H(s)$ are the roots of $P_H(s)$. The uniqueness and stability properties of $P_H(s)$ depend on the location of the roots in the left-hand plane $\Re(s) \leq 0$. The system (1) has a stable and unique solution if the Hurwitz polynomial associated to (2) has n_2 roots larger than zero and n_1 roots lower than zero. To check how many roots are positive and how many are negative in a high order polynomial is a simpler task than to check how many roots are within or outside the unit circle. Moreover, in microfounded macro-models the sign of the parameters defining functional forms is usually known, while their magnitude is not. Hence, verifying if these conditions are satisfied is more straightforward then verifying if BK conditions are.

3. Application: A NK model with nominal rigidities and LAMP

We apply the methodology to a NK model featuring: (i) staggered wage and price contracts; (ii) LAMP, that is, a fraction $\lambda \in [0, 1]$ of agents do not participate to the financial markets and consume their labor income. The model economy is spelled out in Ascari et al. (2017). Log-linear equilibrium dynamics of the model around the efficient steady state are determined by:

$$\begin{aligned}
&(M1) \quad \pi_t = \beta E_t \pi_{t+1} + \kappa_p \tilde{\omega}_t && \textit{NKPC} \\
&(M2) \quad \pi_t^w = \beta E_t \pi_{t+1}^w + \kappa_w [(\sigma+\phi)x_t - \tilde{\omega}_t] && \textit{Wage Inflation Curve} \\
&(M3) \quad \tilde{\omega}_t = \tilde{\omega}_{t-1} + \pi_t^w - \pi_t - \Delta\omega_t^{Eff} && \textit{Real Wage Gap} \\
&(M4) \quad x_t = E_t x_{t+1} - \frac{1}{\sigma} E_t \left(i_t - \pi_{t+1} - r_t^{Eff} \right) \\
&\qquad - \frac{\lambda}{(1-\lambda)} E_t \Delta\tilde{\omega}_{t+1} && \textit{IS curve}.
\end{aligned}$$

Fluctuations are caused by shocks to labor productivity, a_t, and by taste shocks, ψ_t. $(M1)$ is the NKPC, π_t represents deviations of current inflation from its (zero) steady state; $\tilde{\omega}_t = \omega_t - \omega_t^{Eff}$ represents the *real wage gap*, defined as the gap between the current and the efficient real wage. The latter is determined by technology, $\omega_t^{Eff} = a_t$. The slope of the NKPC is $\kappa_p = \frac{(1-\beta\xi_p)(1-\xi_p)}{\xi_p}$, β is the subjective discount factor and ξ_p the Calvo-probability for a firm of not changing its price. The variable π_t^w is wage inflation. The wage inflation curve, $(M2)$, has slope $\kappa_w = \frac{(1-\beta\xi_w)(1-\xi_w)}{\xi_w}$. ξ_w is the Calvo-probability for a labor union of not changing its wage. The variable $x_t = y_t - y_t^{Eff}$ denotes the gap between actual output and the efficient output, which reads as $y_t^{Eff} = \frac{1+\phi}{\sigma+\phi} a_t + \frac{1}{(\sigma+\phi)} \psi_t$. The parameters ϕ and σ are the elasticity of intertemporal substitution in labor supply and in consumption, respectively. $(M3)$ defines the real wage gap. $(M4)$ is the IS curve, which differs from a standard IS equation because of the extra term $\frac{\lambda}{1-\lambda} E_t \Delta\tilde{\omega}_{t+1}$. The expected growth of the real wage affects aggregate demand relative to the efficient allocation through the consumption of constrained consumers — it will not appear if $\lambda = 0$. The efficient real rate of interest is $r_t^{Eff} = \sigma \left(\frac{1+\phi}{\sigma+\phi} \Delta a_{t+1} - \frac{\phi}{\sigma(\sigma+\phi)} \Delta\psi_{t+1} \right)$. We consider the same interest rate rule as in Bilbiie (2008)

$$i_t = \phi_\pi \pi_{t+1}. \tag{4}$$

The relevant system to study the determinacy of REE can be represented as in (1) with $z_t = \left[\pi_t^w, \pi_t, x_t, \tilde{\omega}_t \right]'$ and A being the matrix in Box I.

The 4th-order characteristic polynomial reads as

$$P_C(\gamma) = \gamma^4 + a_1\gamma^3 + a_2\gamma^2 + a_3\gamma + a_4.$$

The latter can be transformed into the Hurwitz polynomial using $\gamma = \frac{1+s}{1-s}$. In this case

$$\tilde{P}_H(s) = \left(\frac{1+s}{1-s}\right)^4 + a_1\left(\frac{1+s}{1-s}\right)^3 + a_2\left(\frac{1+s}{1-s}\right)^2 + a_3\frac{1+s}{1-s} + a_4. \tag{5}$$

Hence one needs to study the following Hurwitz polynomial

$$P_H(s) = \underbrace{\tilde{a}_4}_{\frac{a_1+a_2+a_3+a_4+1}{a_2-a_1-a_3+a_4+1}} + s \underbrace{\tilde{a}_3}_{\frac{2(2+a_1-a_3-2a_4)}{a_2-a_1-a_3+a_4+1}} + s^2 \underbrace{\tilde{a}_2}_{\frac{2(3a_4-a_2+3)}{a_2-a_1-a_3+a_4+1}} + s^3 \underbrace{\tilde{a}_1}_{\frac{2(a_3-a_1-2a_4+2)}{a_2-a_1-a_3+a_4+1}} + s^4. \tag{6}$$

Proposition 1 (*Taylor Principle and LAMP*). *Under policy rule (4) there exists a locally unique rational expectations equilibrium if and only if:*

CASE I: $\lambda < \bar{\lambda}^{FR} : \phi_\pi \in \left(1; \bar{\phi}_\pi^{FR}\right)$;

CASE II: $\lambda > \bar{\lambda}^{FR} : \phi_\pi \in \left(\bar{\phi}_\pi^{FR}; 1\right)$;

where $\bar{\lambda}^{FR} = \frac{1}{1+\frac{\kappa_w(\sigma+\phi)}{2(1+\beta)+\kappa_w+\kappa_p}}$ *and*

$$\bar{\phi}_\pi^{FR} = 1 + \frac{2\sigma(1+\beta)\left[2(1+\beta)+\kappa_p+\kappa_w-\frac{\lambda}{1-\lambda}\kappa_w(\sigma+\phi)\right]}{\kappa_w\kappa_p(\sigma+\phi)}.$$

Case I generalizes the Taylor principle: as in the full-participation case, the central bank should respond more than one-to-one to increases in inflation. Case II corresponds to the Inverted Taylor Principle. In this case, as in Bilbiie (2008), only a passive policy is consistent with a unique REE. While the proof is in Appendix A.1, here we sketch it to show how to apply the suggested methodology. The key point is to adopt the transformation in (6) to go from the characteristic polynomial of matrix A

$$\begin{aligned}
P_C(\gamma) = \gamma^4 &+ \left[\frac{1}{\beta}\left[-2-2\beta-\left(\kappa_w+\kappa_p\right)+\chi\kappa_w(\sigma+\phi)\right]\right]\gamma^3 \\
&+ \left[\frac{1}{\beta}(\kappa_p+\kappa_w+\beta+1) - \frac{1}{\beta}\kappa_w(\sigma+\phi) \right. \\
&\times \left(\chi\left(1+\frac{1}{\beta}\right) + \frac{1}{\sigma\beta}\kappa_p(\phi_\pi-1)\right) \\
&+ \left.\frac{1}{\beta^2}\left(1+3\beta+\kappa_w+\kappa_p\right)\right]\gamma^2 \\
&+ \left[-\frac{1}{\beta^2}\left(2+2\beta+\kappa_w+\kappa_p-\chi\kappa_w(\sigma+\phi)\right)\right]\gamma + \frac{1}{\beta^2}
\end{aligned}$$

to the associated Hurwitz polynomial (see equation given in Box II),

where

$$D = 4\beta^2 + 4 + 8\beta + 2[\beta+1](\kappa_p+\kappa_w) - \frac{1}{\sigma}\kappa_w\kappa_p(\sigma+\phi)(\phi_\pi-1) - 2(1+\beta)\chi\kappa_w(\sigma+\phi).$$

This polynomial should exhibit 3 positive roots and 1 negative root for the REE to be unique. This is an easier condition to check than checking whether 3 roots and 1 root of $P_C(\gamma)$ are outside and inside the unit circle, respectively. The Appendix A analyzes the signs of the coefficients $\tilde{a}_i$, and exploits the Decartes' rule of sign for polynomials.

Besides the methodological aspect, Proposition 1 illustrates the second contribution of the paper. Whenever $\lambda < \bar{\lambda}^{FR}$, the Taylor Principle is necessary and sufficient for determinacy. As in Bilbiie (2008), there is a region of the parameter space where the Taylor principle is inverted: when $\lambda > \bar{\lambda}^{FR}$, ϕ_π needs to be lower than 1 to yield a unique REE.

The value of $\bar{\lambda}^{FR}$ is lower then one, it increases monotonically with the degree of wage stickiness, and tends to one when wages are almost fixed. Fig. 1 depicts the threshold value $\bar{\lambda}^{FR}$ as a function of the degree of wage stickiness, ξ_w. When wages are almost fixed, i.e. when $\xi_w \longrightarrow 1$, then $\kappa_w \longrightarrow 1$ and $\bar{\lambda}^{FR} = \frac{1}{1+\frac{(\sigma+\phi)}{2(1+\beta)+1+\kappa_p}}$. On the contrary, in the case of flexible wages, i.e. $\xi_w \longrightarrow 0$,