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Riskiness in binary gambles: A geometric analysis☆

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HIGHLIGHTS

- Two orderings of binary gambles under riskiness are studied and compared.
- One ordering results from the index of riskiness analyzed by Aumann and Serrano.
- The other ordering arises from the measure of riskiness proposed by Foster and Hart.
- The iso-riskiness curves are defined and their properties analyzed.
- Such curves are represented geometrically for each measure of riskiness.

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ABSTRACT

The orderings of binary gambles under the index of riskiness analyzed by Aumann and Serrano and under the measure of riskiness proposed by Foster and Hart are studied and compared. To proceed the properties of the curves that include gambles with the same level of riskiness are analyzed for each of those measures and such curves are represented geometrically.

1. Introduction

Two measures of riskiness, the index of riskiness R^{AS} characterized by Aumann and Serrano (2008) (hereafter [*AS*]) and the measure of riskiness R^{FH} proposed by Foster and Hart (2009) (hereafter [*FH*]), deserve further analyses, to learn more about how they order gambles and about their differences. From [*AS*] and [*FH*] it is known that R^{AS} and R^{FH} are objective, depend on each moment of the distribution of a gamble, fulfill positive homogeneity, complete, for those gambles in which the corresponding measure exists, the partial ordering given by first and second stochastic dominance and have some other valuable properties.[1]

This work studies R^{AS} and of R^{FH} in binary gambles (risky alternatives) with positive mean and a positive probability of loss. For each of those measures of riskiness the properties of the curves that include gambles with the same level of riskiness are analyzed and such curves are represented geometrically. That analysis allows for a comparison of the orderings of binary gambles by R^{AS} and by R^{FH}.

R^{AS} and R^{FH} are presented in Section 2. Section 3 is devoted to the analysis and comparison of the orderings of binary gambles under R^{AS} and under R^{FH}.

2. Preliminaries

A gamble g is a random variables with positive expectation ($E[g] > 0$) that can take negative values ($P(g < 0) > 0$). The outcomes of a gamble should be understood as net changes to current wealth.

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1 Kadan and Liu (2014) analyze how R^{AS} and R^{FH} depend on each moment of the distribution of outcomes of the gamble.

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Moreover, an expected-utility decision maker with a Bernoulli function u for money is considered. Function u is strictly concave since the decision maker is risk averse.

From [AS] it follows that $R^{AS}(g)$ is the positive solution to[2] :

$$E\left[e^{-\frac{g}{R^{AS}(g)}}\right] = 1, \tag{1}$$

and that its existence is guaranteed for a finitely-valued gamble g if $E[g] > 0$ and $\Pr_g(x < 0) > 0$.[3]

From (1) it emerges that $R^{AS}(g) = \frac{1}{a(g)}$, where $a(g)$ is the coefficient of absolute risk aversion of the CARA decision maker that is indifferent between accepting and rejecting the gamble g, i.e., such that:

$$E\left[e^{-a(g)(w+g)}\right] = e^{-a(g)w} \Leftrightarrow E\left[e^{-a(g)g}\right] = 1.$$

where w is the wealth of the decision maker. Hence, it follows that R^{AS} is independent of w.

[FH] prove that the behavior of a decision maker that accepts a gamble g only if her current wealth is above $R^{FH}(g)$ guarantees no-bankruptcy, that is, guarantees that her wealth will never become negative (in a context in which the decision maker is offered a gamble g_t in every period $t = 0, 1, \ldots$ and has initial wealth w_0). By contrast, bankruptcy may occur if the strategy of the decision maker is such that there are gambles g which would be accepted when her current wealth is below $R^{FH}(g)$. Hence, $R^{FH}(g)$ is the minimal current wealth level at which gamble g may be accepted by the decision maker to guarantee no-bankruptcy. When g is a discrete variable, $R^{FH}(g)$ is higher than minus the maximum loss under g.

In [FH] it is proved that for each finitely-valued gamble g the riskiness $R^{FH}(g)$ is given by[4] :

$$E\left[\log\left(1 + \frac{g}{R^{FH}(g)}\right)\right] = 0. \tag{2}$$

As (2) implies:

$$E\left[\log\left(R^{FH}(g) + g\right)\right] = \log(R^{FH}(g)),$$

it follows that any decision maker should reject at least as much as a decision maker with logarithmic Bernoulli function to avoid bankruptcy.

3. Riskiness in binary gambles

A binary gamble $g \equiv (p, M, L)$ with $P(g < 0) > 0$ is a gamble that may result in a gain of M with probability p or in a loss of L with probability $1 - p$, with $M, L > 0$. As $E[g] > 0$ it follows for gamble g that $pM - (1-p)L > 0 \Rightarrow M > \frac{1-p}{p}L$.

In this section R^{AS} and R^{FH} are analyzed for gambles with the same value of L (the analysis could be replicated for gambles with the same value of M or for gambles with the same value of p). It may be considered that L is the investment required for any of the gambles studied and that the worst result in each gamble implies to lose all that investment. The representations in the (p, M) plane studied below refer to the (p, M) plane that results for the value of L considered, with p measured in the horizontal axis of that plane.

To proceed let us introduce the following definition:

Definition 1. For a measure of riskiness R and binary gambles with the same value of L the iso-R-riskiness of level R is the curve in the (p, M) plane that goes through all the gambles with riskiness R.

There is an iso-R-riskiness in the (p, M) plane for each value of R. Moreover, there is a unique iso-R-riskiness going through each pair (p, M) in the feasible area of that plane (iso-R-riskiness curves do not cut each other).

3.1. AS-riskiness

For the binary gamble (p, M, L) it follows from (1) that $R^{AS} = \frac{1}{a}$ where a is the solution to:

$$pe^{-aM} + (1-p)e^{aL} = 1. \tag{3}$$

From (3) it follows that:

$$M = -\frac{1}{a}\log\left(\frac{1}{p}(1 - (1-p)e^{aL})\right) \tag{4}$$

The straight line $p = 1$ would be the iso-R^{AS}-riskiness of level 0 since for the gambles in that line the probability of loss is 0. Moreover, the curve $M = \frac{1-p}{p}L$ would be the iso-R^{AS}-riskiness of level ∞ as:

$$\lim_{a\to 0} M = \lim_{a\to 0}\left(-\frac{1}{a}\log\left(\frac{1}{p}(1 - (1-p)e^{aL})\right)\right) = \frac{1-p}{p}L.$$

Nevertheless, the straight line $p = 1$ and the curve $M = \frac{1-p}{p}L$ are in the frontier of the open set of gambles in the (p, M) plane that we consider in this analysis of riskiness.

The following is proved, for any fixed value of L:

Proposition 1. *In the (p, M) plane the iso-R^{AS}-riskiness of level R given by (4), with $a = \frac{1}{R}$:*

(i) *is defined only for $p > 1 - e^{-\frac{L}{R}}$,*
(ii) *is decreasing and convex,*
(iii) *is asymptotic to the straight line $p = 1 - e^{-\frac{L}{R}}$,*
(iv) *is on the right of iso-R^{AS}-riskiness of level R' if and only if $R < R'$, and*
(v) *goes through the point ($p = 0, M = 1$) and its slope at that point is lower than $-L$ and less negative the higher is R.*

Proof. (i) As the expression for M in (4) requires that $1 - (1-p)e^{La} = pe^{La} - e^{La} + 1 > 0$ it follows that the iso-R^{AS}-riskiness of level R is defined only for:

$$1 - (1-p)e^{La} > 0 \Leftrightarrow p > 1 - e^{-La} = 1 - e^{-\frac{L}{R}}.$$

(ii) From (4) it follows that the slope of the iso-R^{AS}-riskiness of level $\frac{1}{a}$ in the (M, p) plane, given L, is:

$$\frac{\partial M}{\partial p} = \frac{1 - e^{La}}{ap\left(pe^{La} - e^{La} + 1\right)} < 0 \tag{5}$$

as $pe^{La} - e^{La} + 1 > 0$. Moreover:

$$\frac{\partial^2 M}{\partial p^2} = \frac{(2pe^{La} - e^{La} + 1)(e^{La} - 1)}{ap^2(pe^{La} - e^{La} + 1)^2} > 0.$$

(iii) From (4) and (5) it emerges that:

$$\lim_{p\to 1-e^{-La}} M = \lim_{p\to 1-e^{-La}}\left(-\frac{1}{a}\log\left(\frac{1}{p}(1 - (1-p)e^{La})\right)\right) = \infty$$

and

$$\lim_{p\to 1-e^{-La}} \frac{dM}{dp} = \frac{1}{a}\frac{1 - e^{La}}{\left(1 - e^{-La}\right)\left(e^{La}\left(1 - e^{-La}\right) - e^{La} + 1\right)} = \infty.$$

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[2] See acknowledgments on initial developments and application of R^{AS} in footnote on page 810 of [AS].

[3] For an infinitely-valued gamble h [AS] mention an additional condition: $E(e^{-\alpha h}) < \infty$ for all $\alpha > 0$. Schulze (2014) shows that this latter condition is sufficient, but not necessary, for existence of $R^{AS}(h)$ and obtains a necessary and sufficient condition for the existence of $R^{AS}(h)$.

[4] Riedel and Hellmann (2015) extend R^{FH} to gambles with continuous distributions and a finite maximum loss.