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Cointegration in singular ARMA models

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HIGHLIGHTS

- We consider I(1) ARMA processes with singular error covariance matrix.
- Also in the left coprime case the cointegrating rank shown to depend upon a(1) only.
- Definition and discussion of exact cointegration.

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ABSTRACT

We consider the cointegration properties of singular ARMA processes integrated of order one. Such processes are necessarily cointegrated as opposed to the regular case. We show that in the left coprime case the cointegrating space only depends upon the autoregressive polynomial at one.

1. Introduction

In this paper we analyze the (integration and) cointegration properties of singular AR and ARMA models. Singular AR and ARMA models, i.e., models with singular error variance, occur in the dynamic stochastic general equilibrium (DSGE) literature, if the number of shocks is strictly smaller than the number of observables (see, e.g., Komunjer and Ng, 2011). They also occur in linear dynamic factor models. In this context the latent variables are typically described as a singular AR or ARMA models (see, e.g., Forni et al., 2000). Furthermore, in this setting singular ARMA models may arise as models for the *static factors*, if the number of the static factors is strictly larger than the number of dynamic factors. In a stationary setting singular AR models are treated in Anderson et al. (2012) and the ARMA case is considered in Anderson et al. (2016).

Cointegration properties of singular AR models are analyzed in Barigozzi et al. (2016). Whereas regular AR systems are always left coprime, this is not true for singular AR systems, see Anderson et al. (2012), which thus show a similarity to ARMA models. Taking this similarity into account we are led to considering the cointegration properties of singular ARMA models. We first discuss that in the singular case cointegration is inevitably present. Then we show that in the left coprime case the cointegrating space only depends on the autoregressive polynomial at $z = 1$, as in the regular case.

The paper is organized as follows: Section 2 introduces the setting and states the underlying assumption, Section 3 gives the result and Section 4 briefly concludes.

2. Setting and assumptions

Consider an ARMA system

$$a(z)y_t = b(z)\varepsilon_t, \quad (1)$$

where

$$a(z) := \sum_{j=0}^{p} a_j z^j, \; a_j \in \mathbb{R}^{n\times n}, \; a_0 = I_n,$$

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$$b(z) := \sum_{j=0}^{q} b_j z^j, \;\; b_j \in \mathbb{R}^{n\times q}$$

with z used as both a complex variable as well as the backward shift operator on the integers $\mathbb{Z}$. The process $\{\varepsilon_t\}$ is white noise with $\mathbb{E}\varepsilon_t\varepsilon_t = I_q$. The system is singular if $q < n$ and regular for $q = n$ and $\operatorname{rk} b(z) = q$ for one z.

Assumption 1. Throughout we assume that:

(i) The determinant of the AR polynomial fulfills

$$\det a(z) \neq 0 \text{ for } |z| \leq 1, \text{ except for } z = 1. \tag{2}$$

(ii) System (1) fulfills the strict miniphase assumption

$$\operatorname{rk} b(z) = q \quad |z| \leq 1 \tag{3}$$

(iii) In addition we assume that the system (1) is integrated of order 1, i.e., the transfer function

$$k(z) := a^{-1}(z)b(z) \tag{4}$$

has a pole at $z = 1$, but the function $c(z) := (1 - z)k(z)$ has all poles outside the closed unit circle. Thus,

$$c(z) = \sum_{j=0}^{\infty} c_j z^j \tag{5}$$

is a convergent power series for $|z| \leq 1$.

(iv) The pair $(a(z), b(z))$ is left coprime, i.e., every common left (polynomial matrix) divisor of $a(z)$ and $b(z)$ is a unimodular matrix.

With respect to item (iv) of Assumption 1 note that a polynomial matrix $u(z)$ is unimodular if and only if $\det u(z) \equiv c \neq 0$ and that $(a(z), b(z))$ is left coprime if and only if $\operatorname{rk}\,(a(z), b(z)) = n$ for all $z \in \mathbb{C}$. For a more detailed discussion see, e.g., Hannan and Deistler (2012).

3. The cointegration properties in the singular case

Let

$$\begin{aligned} k(z) &= a^{-1}(z)b(z) \\ &:= u(z)\Lambda(z)v(z) \end{aligned} \tag{6}$$

denote the Smith–McMillan form (see, e.g., Hannan and Deistler, 2012) of the transfer function $k(z)$. Here $u(z)$ and $v(z)$ are unimodular $n \times n$ respectively $q \times q$ matrices and $\Lambda(z)$ is a unique $n \times q$ diagonal matrix of the form

$$\Lambda(z) := \begin{pmatrix} \frac{p_1(z)}{q_1(z)} & & 0 \\ & \ddots & \\ & & \frac{p_q(z)}{q_q(z)} \\ \hline 0 & \cdots & 0 \end{pmatrix}, \tag{7}$$

where $p_i(z), q_i(z)$, for $i = 1, \ldots, q$ are relatively prime and monic (i.e., with leading coefficient equal to one) polynomials, $p_i(z)$ divides $p_{i+1}(z)$ for $i = 1, \ldots, q-1$ and $q_{i+1}(z)$ divides $q_i(z)$ for $i = 1, \ldots, q-1$. For a given transfer function $k(z)$, the zeros of the polynomials $p_i(z)$ are the zeros of $k(z)$ and the zeros of the polynomials $q_i(z)$ are the poles of $k(z)$.

The strict miniphase assumption (3) implies that $p_i(1) \neq 0$ for $i = 1, \ldots, q$ and condition (2) implies that the zeros of $q_i(z)$ are either at $z = 1$ or satisfy $|z| > 1$. Consequently, there is a c, with $1 \leq c \leq q$, such that

$$q_i(z) = (1 - z)\bar{q}_i(z), \quad i = 1, \ldots, c \tag{8}$$

with $\bar{q}_i(1) \neq 0$ for $i = 1, \ldots, c$ and $q_i(1) \neq 0$ for $i = c+1, \ldots, q$. Accordingly we may write

$$\Lambda(z) = \begin{pmatrix} \frac{p_1(z)}{\bar{q}_1(z)}(1-z)^{-1} & 0 & & \cdots & & 0 \\ 0 & \ddots & \ddots & & & 0 \\ & \ddots & \frac{p_c(z)}{\bar{q}_c(z)}(1-z)^{-1} & & & \vdots \\ \vdots & & & \frac{p_{c+1}(z)}{q_{c+1}(z)} & & \\ & & \ddots & & \ddots & 0 \\ 0 & & \cdots & 0 & & \frac{p_q(z)}{q_q(z)} \\ \hline 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}, \tag{9}$$

and

$$c(z) = u(z) \begin{pmatrix} \frac{p_1(z)}{\bar{q}_1(z)} & 0 & & \cdots & & 0 \\ 0 & \ddots & \ddots & & & 0 \\ & \ddots & \frac{p_c(z)}{\bar{q}_c(z)} & & & \vdots \\ \vdots & & & \frac{p_{c+1}(z)}{q_{c+1}(z)}(1-z) & & \\ & & & & \ddots & 0 \\ 0 & & \cdots & 0 & & \frac{p_q(z)}{q_q(z)}(1-z) \\ \hline 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix} \times v(z). \tag{10}$$

Now, consider, a Beveridge–Nelson type decomposition (see, e.g., Beveridge and Nelson, 1981; Phillips and Solo, 1992)

$$c(z) = c(1) + (1 - z)c^*(z), \tag{11}$$

with $c^*(z)$ rational with no poles and zeros inside or on the unit circle.

The solution of the ARMA system on $\mathbb{N}$ we consider (see, e.g., Bauer and Wagner, 2012) is of the form

$$y_t = c(1)\sum_{j=1}^{t} \varepsilon_j + c^*(z)\varepsilon_t. \tag{12}$$

Taking the first difference of y_t as given in (12) leads to a process $y_t - y_{t-1} = \sum_{j=0}^{\infty} c_j \varepsilon_{t-j}$ for $t \geq 2$ that is stationary. As is immediately clear from (12), the left kernel of $c(1)$, $\operatorname{lker} c(1)$, is the space of cointegrating relationships β, say. Clearly, the cointegrating space has at least dimension $n-q$. Thus, in the singular case cointegration is always present. Since $\frac{p_i(1)}{\bar{q}_i(1)} \neq 0$ and both $u(1)$ and $v(1)$ are nonsingular, it is directly seen from (10) that the dimension of $\operatorname{lker} c(1)$ is equal to $n - c$.

Defining

$$\tilde{a}(z) := \begin{pmatrix} \bar{q}_1(z)(1-z) & & & & & & \\ & \ddots & & & & & 0 \\ & & \bar{q}_c(z)(1-z) & & & & \\ & & & q_{c+1}(z) & & & \\ & & & & \ddots & & \\ & & & & & q_q(z) & \\ & & & & & & 1 \\ & 0 & & & & & & \ddots \\ & & & & & & & & 1 \end{pmatrix} \times u^{-1}(z) \tag{13}$$