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# A multi-item auction with budget-constrained bidders and price controls☆

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HIGHLIGHTS

- We consider a multi-item auction model with unit-demand bidders.
- Both budget constraints and price controls are allowed.
- A rationed equilibrium whose allocation is in the core is proposed.
- An ascending auction is constructed to find the proposed equilibrium.

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ABSTRACT

We extend the multi-item auction model of Talman and Yang (2008) and Andersson et al. (2015) by considering both unit-demand bidders with budget constraints and price controls on bidding items. Due to these budget and price restrictions, a Walrasian equilibrium generally fails to exist. To achieve efficiency, we propose a rationed equilibrium whose allocation is in the core. We also construct an ascending auction to find the proposed rationed equilibrium in (pseudo-)polynomial time.



## 1. Introduction

In many allocation problems where money transfers are allowed, such as the allocation of public housing, vehicle licenses and spectrum licenses, the government aims to achieve efficient outcomes, i.e., the items are given to those who value them the most. If bidders are able to pay up to their values of the bidding items and item prices are completely flexible, a Walrasian equilibrium (WE) is well defined, and auctions yielding Walrasian equilibria can be used to solve those allocation problems and achieve efficient outcomes.

However, in reality, *bidders often face budget constraints* and may not be able to afford what the items are worth to them. Budget constraints arise in developing countries and during financial crises, and they are often observed in public asset privatization auctions in Eastern Europe (Maskin, 2000). On the other hand, for political and economic reasons, *item prices are often controlled*, and they are fixed or restricted to some admissible intervals. The best-known examples are the minimum wage acts and rent control policies enacted all over the world (Andersson et al., 2015; Herings, 2015). In either case, a WE generally does not exist, and standard auctions generate *inefficient* outcomes.

Accounting for both budget and price restrictions, we (i) propose a new equilibrium to allocate items as efficiently as possible and (ii) construct an auction to find the proposed equilibrium in a finite number of steps.

We extend the multi-item auction model of Talman and Yang (2008) and Andersson et al. (2015) by considering both unit-demand bidders with budget constraints and price controls on

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bidding items. First, *we propose a rationed equilibrium (RE) and conclude that the RE allocation is a core allocation, thus achieving Pareto efficiency*. Second, *we construct an ascending auction to find the proposed RE in (pseudo-)polynomial time, and thus, a core allocation is also identified.* Without budget and price restrictions, the proposed auction is consistent with the auction of Andersson et al. (2013), so it also identifies a minimum price WE in (pseudo-)polynomial time.

In the auction model with budget-constrained bidders, Talman and Yang (2015) propose a core-selecting auction, but their price-updating process differs from ours. Besides, if bidders do not face budget constraints, their auction does not generally terminate at a minimum price WE. van der Laan and Yang (2016) propose an RE whose allocation may not be in the core; hence, a van der Laan and Yang (2016) auction that converges to their proposed RE does not generally find a core allocation.

In an auction model with price controls, our proposed RE coincides with the RE proposed by Talman and Yang (2008). Our result implies that their proposed RE allocation is a core allocation. Thus, a Talman and Yang (2008) auction that converges to their proposed RE also identifies a core allocation. Andersson et al. (2015) also prove that the auction of Talman and Yang (2008) is a core-selecting auction, but they do not show the connection between the RE allocation and the core allocation.

## 2. The model

An auctioneer wants to sell a finite set of **items**, $M \equiv \{1, 2, \ldots, m\}$, to a finite set of **bidders**, $N \equiv \{1, 2, \ldots, n\}$.[1] Not receiving an item is called receiving a **null** item, item 0. Let $M_0 \equiv M \cup \{0\}$. Each bidder receives at most one item and except for item 0, each item is assigned to at most one bidder. We index the auctioneer by 0 and let $N_0 \equiv N \cup \{0\}$.

Let $\mathbb{N}$ and $\mathbb{N}_+$ be the sets of integer and non-negative integer numbers. For each $i \in N$ and each $x \in M_0$, denote $V^i : M_0 \to \mathbb{N}$ and $b^i_x \in \mathbb{N}_+ \cup \{+\infty\}$ by bidder $i$'s valuation function and budget of item $x$.[2] For each $i \in N$, let $V^i(0) = 0$ and $b^i_0 = 0$. Bidder $i$'s private information is $(V^i(x), b^i_x)_{x\in M}$. Bidder $i$ faces a **budget constraint** if there is $x \in M$ such that $b^i_x < +\infty$.

For each $y \in M_0$, denote $p_y \in \mathbb{N}_+$, $\underline{p}_y \in \mathbb{N}_+$, and $\overline{p}_y \in \mathbb{N}_+ \cup \{+\infty\}$ by item $y$'s price, reserve price (price floor), and price ceiling. Let $0 = \underline{p}_0 = p_0 = \overline{p}_0$. For each $y \in M$, without loss of generality, let $\underline{p}_y = 0$. The price of item $y$ is **controlled** if $\overline{p}_y < +\infty$.

Let $p \equiv (p_0, \ldots, p_m) \in \mathbb{N}^{m+1}_+$ be a price vector. Let $P \equiv \{p \in \mathbb{N}^{m+1}_+ : 0 \le p_y \le \overline{p}_y$ for each $y \in M$ and $0 = \underline{p}_0 = p_0 = \overline{p}_0\}$ be the set of **admissible prices**. For each $i \in N$, let $\pi(i) \in M_0$ be bidder $i$'s assigned item. Let $\pi \equiv (\pi(1), \ldots, \pi(n)) \in (M_0)^n$ be an **item assignment** such that for each pair $i, j \in N$, if $\pi(i) \ne 0$ and $i \ne j$, then $\pi(i) \ne \pi(j)$. Denote the set of item assignments by $\Pi$. A **feasible allocation** is a pair $(\pi, p) \in \Pi \times P$ such that for each $i \in N$, $p_{\pi(i)} \le \min\{b^i_{\pi(i)}, \overline{p}_{\pi(i)}\}$.

Given a pair $(\pi, p) \in \Pi \times P$, the auctioneer's utility is given by $U^0(\pi, p) = \sum_{i\in N} p_{\pi(i)}$, and for each $i \in N$, bidder $i$'s utility is given by

$$U^i(\pi, p) \equiv U^i((\pi, p), (V^i(x), b^i_x)_{x\in M}) = \begin{cases} V^i(\pi(i)) - p_{\pi(i)} & \text{if } p_{\pi(i)} \le b^i_{\pi(i)} \\ -\infty & \text{otherwise} \end{cases}.$$

**Remark 1.** The model reduces to (a) the classical assignment model studied by, e.g., Shapley and Shubik (1971), Demange et al. (1986), and Andersson et al. (2013), if for each $i \in N$ and each $x \in M$, $b^i_x = +\infty$ and $\overline{p}_x = +\infty$; (b) the auction model with budget-constrained bidders studied by, e.g., Lavi and May (2012), Talman and Yang (2015), and van der Laan and Yang (2016), if for each pair $x, y \in M$ and each $i \in N$, $\overline{p}_x = \overline{p}_y = +\infty$ and $b^i_x = b^i_y < +\infty$; (c) the auction model with price controls studied by, e.g., Talman and Yang (2008) and Andersson et al. (2015), if for each $x \in M$ and each $i \in N$, $b^i_x = +\infty$ and $\overline{p}_x < +\infty$.

## 3. Definitions of equilibria

Let bidder $i$'s **demand set** at $p \in P$ be $D_i(p) \equiv \{x \in M_0 : p_x \le b^i_x$ and $V^i(x) - p_x \ge V^i(y) - p_y$ for each $y \in M_0$ such that $p_y \le b^i_y\}$.

**Definition 1.** A pair $(\pi, p) \in \Pi \times P$ is a **Walrasian equilibrium (WE)** if
(i) for each $i \in N$, $\pi(i) \in D_i(p)$;
(ii) for each $y \in M$, if for each $i \in N$, $\pi(i) \ne y$, then $p_y = 0$.

An item assignment $\pi \in \Pi$ is **socially efficient** if for each $\pi' \in \Pi$, $\sum_{i\in N} V^i(\pi(i)) \ge \sum_{i\in N} V^i(\pi'(i))$. In our benchmark case, bidders do not face budget constraints, and there are no price controls.

**Fact 1** (*Shapley and Shubik, 1971*). Assume that for each $i \in N$ and each $x \in M$, $b^i_x = +\infty$ and $\overline{p}_x = +\infty$. Then, (i) a WE exists; (ii) the set of WE prices is a complete lattice, and a minimum WE price uniquely exists; and (iii) any WE item assignment is socially efficient.

If the assumption in Fact 1 is violated, Fact 1 generally fails to hold.

**Fact 2** (*Talman and Yang, 2008; van der Laan and Yang, 2016*). Assume that (a) there is $i \in N$ and $x \in M$ such that $b^i_x < +\infty$ or that (b) there is $y \in M$ such that $\overline{p}_y < +\infty$. Then, a WE is generally not well defined.

The disadvantages of WE motivates us to introduce a new equilibrium concept. For each $i \in N$ and each $x \in M_0$, let $R^i_x \in \{0, 1\}$ be bidder $i$'s rationed scheme over item $x$, namely, $R^i_x = 1$ means that bidder $i$ is allowed to demand $x$, while $R^i_x = 0$ means that bidder $i$ is not allowed to demand $x$. For each $i \in N$, let $R^i_0 = 1$. Let $R^i \equiv (R^i_0, \ldots, R^i_m) \in \{0, 1\}^{m+1}$ be bidder $i$'s rationed scheme and $R \equiv (R_i)_{i\in N} \in [\{0, 1\}^{m+1}]^n$ be a rationed scheme profile. Let bidder $i$'s **rationed demand set** at $p \in P$ be $D_i(p, R^i) \equiv \{x \in M_0 : R^i_x = 1, p_x \le b^i_x$, and $V^i(x) - p_x \ge V^i(y) - p_y$ for each $y \in M_0$ such that $R^i_y = 1$ and $p_y \le b^i_y\}$.

**Definition 2.** A tuple $((\pi, p), R) \in \Pi \times P \times [\{0, 1\}^{m+1}]^n$ is a **rationed equilibrium (RE)** if
(i) for each $i \in N$, $\pi(i) \in D_i(p, R^i)$;
(ii) for each $y \in M$, if for each $i \in N$, $\pi(i) \ne y$, then $p_y = 0$;
(iii) if there is $i \in N$ and $x \in M$ such that $R^i_x = 0$, then
(iii-1) there is $j \in N \setminus \{i\}$ such that $\pi(j) = x$,
(iii-2) $p_x = \min\{b^i_x, \overline{p}_x\}$, and
(iii-3) if $\widetilde{R}^i_x = 1$, then $x \in D_i(p, (\widetilde{R}^i_x, R^i_{-x}))$.

An RE consists of an allocation $(\pi, p)$ and a rationed scheme profile $R$. If for each $i \in N$ and each $x \in M$ we have $R^i_x = 1$, then the RE allocation is a WE.

The RE item assignment $\pi$ may not be socially efficient.[3] However, the RE allocation $(\pi, p)$ is a core allocation, thus achieving Pareto efficiency.[4]

[1] Items can be identical, heterogeneous, or both.

[2] Chen et al. (2010) also study an auction model with item-wise budget-constrained bidders.

[3] A numerical example is provided in a supplementary note (see Appendix A).

[4] A feasible allocation $(x, p) \in \Pi \times P$ is Pareto efficient if there is no feasible allocation $(x', p') \in \Pi \times P$ such that for each $i \in N_0$, $U^i(x', p') > U^i(x, p)$.