Economics Letters 152 (2017) 57–61

Contents lists available at ScienceDirect

Economics Letters

journal homepage: www.elsevier.com/locate/ecolet

Non-existence of optimal programs for undiscounted growth models in continuous time

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HIGHLIGHTS

- We prove that the minimum dimension for non-existence of optima in continuous time is 2.
- This confirms a conjecture advanced in 1976 by Brock and Haurie.
- We work in the framework of a two-dimensional optimal growth model à la Bruno (1967).

ARTICLE INFO

Article history:
Received 14 September 2016
Received in revised form 8 December 2016
Accepted 16 December 2016
Available online 24 December 2016

JEL classification:
C61
D90

Keywords:
Optimal growth
Overtaking
Continuous time models

ABSTRACT

We report an example of a two-dimensional undiscounted convex optimal growth model in continuous time in which, although there is a unique "golden rule", no overtaking optimal solutions exists in a full neighborhood of the steady state. The example proves, for optimal growth models, a conjecture advanced in 1976 by Brock and Haurie that the minimum dimension for non-existence of overtaking optimal programs in continuous time is 2.

1. Introduction

For the class of undiscounted convex models of optimal growth, it has been known since (Gale, 1967) that existence of optimal (in the sense of overtaking) solutions cannot be proved in general if the "golden rule" capital stock is not unique. Soon, however, it turned out that uniqueness is not sufficient for the existence of an optimal solution. (Brock, 1970), indeed, proved existence under this condition, but used the weaker optimality criterion know as maximality (or weak overtaking optimality) and presented an example of a maximal steady state that is not optimal. Peleg (1973) then pointed out that the same example can be used to prove non-existence of optimal paths, implying that, without additional assumptions, it is not possible to strength Brock's existence theorem.

There are only few published examples of non-existence: the Brock–Peleg one, one reported in Khan and Piazza (2010), one contained in a paper by Leizarowitz (1985) and finally the one provided in a paper by Fabbri et al. (2015). While the first two relate to different two-sector one capital good discrete models, the last two are in continuous time. Still, the two-dimensional (Leizarowitz, 1985) example is framed in reduced form, while that in Fabbri et al. (2015), explicitly relating to a growth model, has an infinite-dimensional state space. So, while it has been already established that in discrete time non-existence is possible even with a one-dimensional state space, it is not clear which is the minimum dimension for non-existence for continuous time models.[1] We here report a new example showing that this minimum dimension is 2. In other words, our example confirms the conjecture advanced in Brock and Haurie (1976) p. 345 for optimal growth models:

> *We have not yet constructed an example where the steady state $\bar{x}$ is unique but no overtaking optimal program exists from some x^0 while a weakly overtaking optimal program exists from our x^0.*

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[1] It is known (see e.g. Example 4.1 of Leizarowitz, 1985 or Example 4.4 of Carlson et al., 1991) that optimal trajectories exist for continuous time scalar systems.

http://dx.doi.org/10.1016/j.econlet.2016.12.022
0165-1765/

Such an example will take some work to construct because it seems that the state space will have to be two dimensional whereas in discrete time as shown in Brock (1970) *we can get by with a one-dimensional output space.*

2. The model

We consider the $(n+1)$-sector single-technique case of the discrete capital model introduced in Bruno (1967). In the system, there are $n+1$ commodities: n pure capital goods and a pure consumption good. The services of a primary factor of production, labor, are combined with the services of the stocks of capital to produce the $n+1$ commodities. Technology is of the discrete type, and only $n+1$ processes, one for each good, are available.

The superscript T denotes transposed matrices, $\langle\cdot,\cdot\rangle$ represents the internal product in $\mathbb{R}^n$. A unit of the jth capital good needs to be produced a_{ij} units of the ith capital good and ℓ_j units of labor, whereas one unit of the consumption good needs α_i units of the ith capital good and ℓ_c units of labor, so that the technology is described by a matrix and a vector of capital coefficients $A = \left[a_{ij}\right]_{i,j=1}^n$, $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_n]^T$, and a vector and a scalar of labor input coefficients $\ell = [\ell_1, \ell_2, \ldots, \ell_n]^T$, ℓ_c. Let $k(t) = [k_1(t), k_2(t), \ldots, k_n(t)]^T$ represent the stock of capital goods at a given time $t \geq 0$, and $x(t) = [x_1(t), x_2(t); \ldots, x_n(t)]^T$, and $x_c(t)$ be the intensities of activation of the production processes at that time, chosen by the social planner. Assuming that the flow of new capitals is accumulated and that capitals decay at a constant depreciation rate $\delta > 0$ (the same for all capital goods), and that the initial state of the system is $k_0 \geq 0$, then the state equation is given by the n-dimensional system

$$\dot{k}(t) = -\delta k(t) + x(t), \ t \geq 0; \quad k(0) = k_0. \tag{1}$$

Assume that the labor flow available at every t is constant and normalized to 1, and that every unit of capital good instantaneously provides one unit of production services. Then the production is subject to the following set of constraints, holding for all $t \geq 0$:

$$Ax(t) + x_c(t)\alpha \leq k(t), \tag{2}$$

$$\langle \ell, x(t)\rangle + x_c(t)\ell_c \leq 1, \tag{3}$$

$$x(t) \geq 0, x_c(t) \geq 0. \tag{4}$$

Assuming a linear utility and a discount factor $\rho \geq 0$, the problem is that of maximizing

$$J(x, x_c, k_0) = \int_0^{+\infty} e^{-\rho t} x_c(t)\,dt \tag{5}$$

over the set of admissible controls

$$\mathcal{X}(k_0) = \{(x, x_c) \in L^1_{loc}(0, +\infty; \mathbb{R}^{n+1}_+) \ : \ (1)\text{–}(4) \text{ hold at all } t \geq 0\}.$$

Remark 2.1. Since from (1) one derives $k(t) = e^{-\delta t}k_0 + \int_0^t e^{-\delta(t-s)}x(s)ds$, the solution k is in the space $W^{1,1}_{loc}(0, +\infty; \mathbb{R}^n)$, and trajectories k are always nonnegative. Moreover, if vector ℓ is strictly positive, we may define $c := \left(\sum_{i=1}^n \ell_i^{-2}\right)^{1/2}$ and check that $\|k(t)\| \leq \|k_0\| + c/\delta, \ \forall t \geq 0$, that is, trajectories are uniformly bounded by a constant depending only on k_0. □

Due to (3) and (4), when $\rho > 0$ the utility is finite for all admissible controls but, on the contrary, when $\rho = 0$ it may be infinite valued. We take into consideration the following criteria of optimality.

Definition 2.2. A control $\left(x^*, x_c^*\right)$ in $\mathcal{X}(k_0)$ is optimal (or overtaking optimal) at k_0 if

$$\liminf_{T\to+\infty} \int_0^T e^{-\rho t}(x_c^*(t) - x_c(t))\,dt \geq 0$$

for every control (x, x_c) in $\mathcal{X}(k_0)$. If k^* is the trajectory starting at k_0 and associated to (x^*, x_c^*), then $(k^*; (x^*, x_c^*))$ is an optimal couple.

Definition 2.3. A control $\left(x^*, x_c^*\right)$ in $\mathcal{X}(k_0)$ is maximal (or weakly overtaking optimal) at k_0 if

$$\limsup_{T\to+\infty} \int_0^T e^{-\rho t}(x_c^*(t) - x_c(t))\,dt \geq 0$$

for every control (x, x_c) in $\mathcal{X}(k_0)$. If k^* is the trajectory starting at k_0 and associated to (x^*, x_c^*), then $(k^*; (x^*, x_c^*))$ is a maximal couple.

Every optimal control is maximal but the vice versa is false in general.

We here list the assumptions that will be used throughout the paper.

Hypothesis 2.4.

(1) The matrix A is semipositive, that is, $a_{ij} \geq 0$ for all i and j and there is at least a strictly positive element;
(2) The vector α is semipositive, that is, $\alpha \geq 0$ and $\alpha_i > 0$ for at least one i.
(3) The vector ℓ is positive, that is, $\ell_i > 0$ for all i; also $\ell_c > 0$.
(4) A is indecomposable.

For indecomposable technologies see e.g. Section A.3.2 in Kurz and Salvadori (1995).

2.1. Golden rules

The aim of this section is to define *golden rules*, that is, stationary solutions supported by stationary prices. Some properties of Hamiltonian functions will prove useful for the arguments developed afterwards. We define the *current value Hamiltonian* associated to the problem as the function $h : \mathbb{R}^n_+ \times \mathbb{R}_+ \times \mathbb{R}^n_+ \times \mathbb{R}_+ \to \mathbb{R}$ given by $h(k, \lambda, x, x_c) = x_c + \langle \lambda, x - \delta k\rangle$ and the *maximal value Hamiltonian* as

$$H(k, \lambda) = \sup\{h(k, \lambda, x, x_c) \ : \ (x, x_c) \geq 0, Ax + x_c\alpha \leq k, \langle \ell, x\rangle + x_c\ell_c \leq 1\}. \tag{6}$$

The maximization process through which H is computed, corresponds to solving the following linear programming problem

$$\max[\langle \lambda, x\rangle + x_c] \tag{7}$$

subject to

$$Ax + x_c\alpha \leq k, \quad \langle \ell, x\rangle + x_c\ell_c \leq 1, \quad (x, x_c) \geq 0. \tag{8}$$

which has feasible region

$$U(k) = \{(x, x_c) \in \mathbb{R}^n_+ \times \mathbb{R}_+ \ : \ (8) \text{ holds}\}.$$

The corresponding dual problem is

$$\min[\langle q, k\rangle + w] \tag{9}$$

subject to

$$\lambda \leq A^T q + w\,\ell, \quad 1 \leq \langle \alpha, q\rangle + w\,\ell_c, \quad q \geq 0, w \geq 0, \tag{10}$$

where $(q, w) \in \mathbb{R}^n \times \mathbb{R}$ are dual control variables having the meaning, respectively, of rental rates of capital goods and wage rate (i.e., the multiplier associated to the constraint of availability of labor). We denote the feasible region of the dual problem by

$$V(\lambda) = \{(q, w) \in \mathbb{R}^n_+ \times \mathbb{R}_+ \ : \ (10) \text{ holds}\}.$$