Perturbed best response dynamics in a hawk–dove game

Volker Benndorf [a], Ismael Martínez-Martínez [b],*

[a] *Goethe University Frankfurt, Theodor-W.-Adorno-Platz 4, 60323 Frankfurt am Main, Germany*
[b] *Düsseldorf Institute for Competition Economics (DICE), Heinrich-Heine-Universität Düsseldorf, Universitätsstraße 1, 40225 Düsseldorf, Germany*

ARTICLE INFO

ABSTRACT

We examine the impact of behavioral noise on equilibrium selection in a hawk–dove game with a model that linearly interpolates between the one- and two-population structures in an evolutionary context. Perturbed best response dynamics generates two hypotheses in addition to the bifurcation predicted by standard replicator dynamics. First, when replicator dynamics suggests mixing behavior (close to the one-population model), there will be a bias against hawkish play. Second, polarizing behavior as predicted by replicator dynamics in the vicinity of the two-population model will be less extreme in the presence of behavioral noise. We find both effects in our data set.

1. Introduction

Evolutionary models provide key insights for the understanding of central aspects of strategic interactions. For instance, the *mass-action* interpretation of mixed Nash equilibria emerges naturally in the evolutionary context (Björnerstedt and Weibull, 1996; Young, 2011). The aggregate strategy of a population can be interpreted as a mixed strategy even though each individual agent chooses a pure strategy. Equilibrium selection is another cornerstone of these models. It is possible to discern whether an equilibrium is more or less likely to be selected depending on the structure of the population (Friedman, 1991; Weibull, 1995).

The family of *perturbed best response dynamics* (PBR) can take into account departures from the best-response paradigm, which is not possible with the standard model of replicator dynamics. The PBR models introduce a random component in the definition of the best-response correspondences. Players are assumed to behave as myopic best responders—just as in replicator dynamics—but with the additional feature that they may tremble in their decisions (Blume, 1993). In the limit where the impact of the error term approaches zero, the PBR predictions converge toward those of replicator dynamics. However, for moderate levels of noise, PBR

models can account for some behavioral patterns that deviate from the standard approach (see Hofbauer and Hopkins, 2005; Hofbauer and Sandholm, 2002; Hopkins, 2002 for technical details).

In this note, we apply a PBR model to a hawk–dove game. The hawk–dove game is a symmetric two-strategy game with three Nash equilibria: a symmetric one in mixed strategies and two asymmetric ones in pure strategies.

There are two common (and simplified) ways to implement this interaction as a population game. In the *one-population* case, the interaction only takes place between agents *within* the groups, while in the *two-population* case, the interaction occurs exclusively *between* the groups. Basic intuition in population games argues that mixing behavior emerges when the game is played within the population (one-population matching) because only symmetric equilibria can survive. The polarized case is more likely to be observed in the two-population matching (Oprea et al., 2011).

A recent experiment by Benndorf et al. (2016) relaxes the assumptions for the matching discussed above by introducing a *coupled* model. This allows to linearly interpolate between both extreme structures (one- and two-population models) with a coupling parameter $\kappa \in [0, 1]$. This parameter is a measure for the relative importance of the interaction between the populations. In a discrete analogy, it can also be interpreted as the probability that an agent is matched with an agent from the other group. Note that $\kappa = 0$ and $\kappa = 1$ correspond to the one- and two-population cases, respectively. The coupled model unveils the transition regime from

symmetric mixing to polarized behavior in pure strategies. Given the payoff parameters in the experiment, replicator dynamics predicts symmetric mixed play with $\frac{2}{3}$ of hawk for $\kappa < \frac{1}{2}$. A sudden bifurcation occurs at $\kappa = \frac{1}{2}$ such that one population plays pure hawk and the other plays a mixed strategy with $\frac{1}{3}$ of hawk. Separation (difference in the share of hawk play) increases monotonically with κ in the interval $\kappa \in [\frac{1}{2}, \frac{2}{3}]$. Finally, the system is fully polarized for $\kappa > \frac{2}{3}$. The experiment by Benndorf et al. (2016) largely confirms these predictions, but the authors also report some subtle discrepancies between the data and the replicator model.

In the present paper, we complement their analysis with the study of logit response dynamics as a natural extension of the standard replicator model.[1] This is a common implementation of a PBR model and assumes that the random component follows the logistic distribution. The PBR model applied to our experimental setting makes two predictions that go beyond the scope of the best-response paradigm of replicator dynamics. First, the share of hawk choices in the symmetric mixed equilibrium will be lower than $\frac{2}{3}$. Second, PBR implies that the impact of the polarizing forces on the behavior of the system will be weaker than suggested by replicator dynamics. This effect has two interpretations (see further explanation of the model below). The separation between the two populations will be lower than predicted by replicator dynamics. An alternative perspective is that the value of the coupling parameter κ for which the system transits from the mixed regime to the asymmetric configuration will be higher than $\frac{1}{2}$. This noisy decision rule accounts for the discrepancies between the standard replicator predictions and the behavioral patterns observed in the experiment.

2. PBR model

We consider two populations of players (X and Y) in a two-strategy environment. Let $S^X = \{(s_1, s_2) : s_1^X + s_2^X = 1\}$ such that any point in it represents the share of each strategy among population X (equivalent definition for population Y). The pair (x, y) gives the state of the system with $x = s_1^X$ and $y = s_1^Y$. Then, s_2^X and s_2^Y are given by $1 - x$ and $1 - y$, respectively.

We interpolate the play of the game between the one- and the two-population models with a coupling parameter $\kappa \in [0, 1]$. Recall that when $\kappa = 0$, a player only participates in interactions within her own population. If $\kappa = 1$, the player interacts only with the agents of the other population. Intermediate values of κ correspond to simultaneous interactions at the intra- and intergroup level (Benndorf et al., 2016, Section 3). The instantaneous payoff earned by a player in population X choosing strategy s_i for a given state of the system (x, y) is $\pi_X(s_i; x, y) = (1 - \kappa)[\pi_{i1}x + \pi_{i2}(1 - x)] + \kappa[\pi_{i1}y + \pi_{i2}(1 - y)]$ where π_{ij} are the elements of a 2×2 payoff matrix.

According to the logit response function, a player in population X who observes a choice profile in the populations (x, y), and given the chance to revise the play, chooses action s_1 with probability

$$p_X(s_1; x, y) = \frac{1}{1 + e^{-\lambda \Delta \pi_X(x, y)}}. \tag{1}$$

$\Delta \pi_X(x, y) = \pi_X(s_1; x, y) - \pi_X(s_2; x, y)$ is the payoff advantage (in population X) of strategy s_1 over strategy s_2. Analogous for Y. The comparison of profits influences the dynamics of the system weighted by $\lambda \in [0, \infty)$. This parameter captures deviations from the best response function. If $\lambda = 0$, the revision mechanism is independent from the payoff structure of the game and the system

evolves toward an equal share of strategies in the populations. When $\lambda \to \infty$, PBR approaches replicator dynamics.

We define the action set $S = \{s_1, s_2\}$ such that s_1 corresponds to strategy hawk, and s_2 to dove. Then, the hawk–dove game in matrix notation is

$$\Pi = \begin{pmatrix} a + \frac{1}{2}(v - c) & a + v \\ a & a + \frac{1}{2}v \end{pmatrix}. \tag{2}$$

This game represents a conflict of cost c over a scarce resource of value $0 < v < c$, and $a > 0$ is an endowment of the players. With these parameters (and the payoff function above) we obtain the fitness function $\Delta \pi_X(x, y) = \frac{1}{2}[v - c(x + \kappa(y - x))]$. $\Delta \pi_Y$ is defined analogously.

The logit response dynamics is given by the following system of coupled differential equations:

$$\begin{cases} \dot{x} = p_X(s_1; x, y) - x \\ \dot{y} = p_Y(s_1; x, y) - y, \end{cases} \tag{3}$$

with p_X and p_Y defined in (1). A rest point of (3) corresponds to the logit quantal response equilibrium (McKelvey and Palfrey, 1995) for the given value of the parameter λ.

We illustrate the predictions of the PBR model in Fig. 1. Panel (a) contains several cuts of the stable manifold of (3) for different values of the rationality parameter λ that show the shape of the bifurcation as a function of κ. The prediction for $\lambda = 0$ is independent of the coupling condition and corresponds to uniform randomization. When λ increases, the bifurcation diagram of the system becomes closer to the prediction with replicator dynamics the higher the value of λ.

For every sufficiently high value of the exponent λ (representing low levels of noise in the best response correspondences of the players), there exists a critical value κ_{crit} such that the equilibrium stability shifts from the mixed configuration toward a polarized one. We compute κ_{crit} as a function of λ in panel (b). This value converges monotonically toward $\frac{1}{2}$ when the PBR model degenerates in the replicator dynamics ($\lambda \to \infty$). The same logic applies to the share of the hawk choices in the populations for the regime with low coupling ($\kappa < \kappa_{\text{crit}}$). We illustrate in panel (c) how the level of hawk play monotonically increases with λ and converges to the mixed NE, $v/c = \frac{2}{3}$, when $\lambda \to \infty$.

From this discussion, we see that the PBR model generates two testable hypotheses about human behavior in the experiment:

H1. The share of hawk choices in the populations X and Y for treatments with $\kappa < \frac{1}{2}$ will be lower than $v/c = \frac{2}{3}$ and higher than $\frac{1}{2}$.

H2. The observed separation between populations (difference between hawk play in groups X and Y) for the treatment with $\kappa = 0.6$ will be lower than $\frac{5}{6}$.

The first hypothesis mirrors regular findings regarding mixing behavior in the quantal response literature (Goeree et al., 2016), but the second formulation deserves some explanation. Replicator dynamics makes a sharp prediction concerning the location of the splitting point ($\kappa_{\text{crit}} = \frac{1}{2}$). By contrast, the presence of noise in the best response function shifts the location of the critical level of coupling κ_{crit} for which polarization begins. In the noisy model, this point is generally higher than $\frac{1}{2}$ for low levels of λ (more noise) and decreases monotonically toward the replicator prediction as $\lambda \to \infty$. It is not possible to make an ex-ante point prediction for λ and one cannot cover all possible values of κ as a treatment variable. Therefore, the exact point κ_{crit} cannot be directly observed in an experiment; however, we can still identify the effect of the possible upward shift of such a splitting point. For this, we measure the separation between groups for $\kappa = 0.6$ (the first one that we

---

[1] Traulsen et al. (2010) provide evidence supporting this method of strategy updating in human behavior. Alós-Ferrer and Netzer (2010) and Zhuang et al. (2014) characterize some theoretical long run properties of the model.