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On the memory of products of long range dependent time series[☆]

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HIGHLIGHTS

- This paper derives the memory of the product of two Gaussian fractionally integrated processes.
- Products of fractionally cointegrated series and squared series are also considered.
- It is found that the transmission of memory from the factor series to the product series depends critically on the means of the processes.
- A Monte Carlo simulation confirms the results.
- Implications of the findings for random coefficient models and time series regressions are discussed.

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ABSTRACT

This paper derives the memory of the product series $x_t y_t$, where x_t and y_t are stationary long memory time series of orders d_x and d_y, respectively. Special attention is paid to the case of squared series and products of series driven by a common stochastic factor. It is found that the memory of products of series with non-zero means is determined by the maximal memory of the factor series, whereas the memory is reduced if the series are mean zero.

1. Introduction

Products of time series are at the heart of many non-linear models used in modern finance, examples include the conditional CAPM with time-varying beta, multiplicative component GARCH models, multiplicative error models or time-varying factor models, cf. for example and Ghysels (1998), Engle and Sokalska (2012), Brownlees et al. (2012) and Bollerslev and Zhang (2003). Since it is well known that there is long memory in many economic and financial time series – such as realized volatilities or inflation rates – it is of interest how long range dependence is translated from the factor series x_t and y_t to the product series $z_t = x_t y_t$. In this paper, it is shown that the transmission of memory critically depends on the means of the processes. While the memory of products is the maximum of the memory orders of the factor series if the means are non-zero, the memory order in the product series will be reduced for zero-mean processes.

In a related literature Granger and Hallman (1991) and Corradi (1995) have studied the properties of non-linear transformations of integrated variables. For Gaussian long memory time series Dittmann and Granger (2002) have derived the memory properties for transformations of zero-mean time series, if the transformation can be expressed as a finite sum of Hermite polynomials. A general method to derive the autocovariance function of non-linear transformations of time series that does not require normality is proposed by Abadir and Talmain (2005). The memory of products of long memory time series, however, has not been covered.[2]

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2 Note that the application of a log-transformation does not mitigate this issue. It merely converts the problem into determining the memory of the sum of non-linearly transformed series, but the logarithm cannot be represented as required by Dittmann and Granger (2002). Therefore, the memory of the log-transformed series is unknown—as is that of the product.

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The autocovariance function of the product of two weakly stationary time series x_t and y_t was derived by Wecker (1978). If both series are Gaussian, it is given by

$$\gamma_{xy}(\tau) = \mu_x^2\gamma_y(\tau) + \mu_y^2\gamma_x(\tau) + \mu_x\mu_y[\xi(\tau) + \xi(-\tau)] \\ + \gamma_x(\tau)\gamma_y(\tau) + \xi(\tau)\xi(-\tau), \quad (1)$$

where $\xi(\tau)$ denotes the cross-covariance function at lag τ defined as $\xi(\tau) = E[(x_t - \mu_x)(y_{t-\tau} - \mu_y)]$ and μ_a denotes the expectation of the respective series a_t.

In the remainder of this paper, the memory properties of the product series z_t will be derived from the asymptotic behavior of (1), as $\tau \to \infty$. Definitions, assumptions and the main result are given in Section 2. Sections 3 and 4 extend these results to squares of long memory series and products of variables with common long range dependent factors. Section 5 presents some Monte Carlo results. Conclusions are drawn in Section 6.

2. Persistence of products of long memory time series

In the following, a time series x_t is a long memory series with parameter d_x if its spectral density $f_x(\lambda)$ at frequency λ obeys the power law

$$f_x(\lambda) \sim g_x(\lambda)\lambda^{-2d_x}, \quad (2)$$

as $\lambda \to 0_+$, or if its autocovariance function $\gamma_x(\tau)$ at lag τ is

$$\gamma_x(\tau) \sim G_x(\lambda)\tau^{2d_x-1}, \quad (3)$$

for $\tau \to \infty$. Here, $g_x(\lambda)$ and $G_x(\lambda)$ denote functions that are slowly varying at zero and infinity, respectively. As Beran et al. (2013) show, these definitions are equivalent under fairly general conditions. Hereafter, we write $x_t \sim LM(d_x)$ if x_t fulfills at least one of (2) or (3). For simplicity, we will treat g_x and G_x as constants. The properties of any x_t that is $LM(d_x)$ depend on the value of $d_x \in (-1/2, 1/2)$. For $d_x < 0$, the process is antipersistent, and $f_x(0) = 0$. If $d_x = 0$, $f_x(0) = g_x$ and the process has short memory. Finally, for $d_x > 0$, x_t is long range dependent.

Here, we follow Dittmann and Granger (2002) and distinguish between fractional integration and long memory. The reason is, that we derive the memory of $z_t = x_t y_t$ based on the behavior of $\gamma_{xy}(\tau)$ for large τ that is of the form specified in (3), so that its spectral density is of the form given in (2). A fractionally integrated process $\tilde{z}_t$, on the other hand, has spectral density $f_{\tilde{z}}(\lambda) = |1 - e^{i\lambda}|^{-2d_{\tilde{z}}} g_{\tilde{z}}(\lambda)$, so that $f_{\tilde{z}}(\lambda) \sim g_{\tilde{z}}|\lambda|^{-2d_{\tilde{z}}}$, as $\lambda \to 0_+$, since $|1-e^{i\lambda}| \to \lambda$, as $\lambda \to 0_+$. While fractional integration is therefore a special case of long memory, the results given here only allow to draw conclusions about the memory properties of the product series.

For the main result we require the following assumptions.

Assumption 1. $x_t \sim LM(d_x)$ and $y_t \sim LM(d_y)$ are weakly stationary and causal Gaussian processes, with $0 \leq d_x, d_y < 0.5$ and finite second order moments.

Assumption 2. If $x_t, y_t \sim LM(d)$, then $x_t - \psi_0 - \psi_1 y_t \sim LM(d)$ for all $\psi_0, \psi_1 \in \mathbb{R}$.

Assumption 1 is a simple regularity condition, whereas Assumption 2 precludes the presence of fractional cointegration. This will be relaxed in Section 4, where the case of a common long memory factor driving x_t and y_t is considered.

Given these assumptions, the memory of the product series $x_t y_t$ is characterized by the following proposition.

Proposition 1. *Under Assumptions 1 and 2 the product series $z_t = x_t y_t$ is $LM(d_z)$, with*

$$d_z = \begin{cases} \max(d_x, d_y), & \text{for } \mu_x, \mu_y \neq 0 \\ d_x, & \text{for } \mu_x = 0, \mu_y \neq 0 \\ d_y, & \text{for } \mu_y = 0, \mu_x \neq 0 \\ \max\left\{d_x + d_y - 1/2, 0\right\}, & \text{for } \mu_y = \mu_x = 0 \text{ and } S_{xy} \neq 0 \\ d_x + d_y - 1/2, & \text{for } \mu_y = \mu_x = 0 \text{ and } S_{xy} = 0, \end{cases}$$

where $S_{xy} = \sum_{\tau=-\infty}^{\infty} \gamma_x(\tau)\gamma_y(\tau)$.

Proof. The autocovariance function of any $x_t y_t$ satisfying Assumption 1 is given by (1). This is a linear combination of the autocovariance functions $\gamma_x(\tau)$, $\gamma_y(\tau)$, the cross-covariance function $\xi(\tau)$ and interaction terms between them. Since long memory is defined in (3) by the shape of the autocovariance function for $\tau \to \infty$, we can determine the memory of $x_t y_t$ by finding the limit of $\gamma_{xy}(\tau)$. For $\tau \to \infty$, we can substitute $\gamma_x(\tau)$ and $\gamma_y(\tau)$ with $G_x\tau^{2d_x-1}$ and $G_y\tau^{2d_y-1}$ from (3). The asymptotic properties of the cross-covariance function $\xi(\tau)$ can be derived from results of Phillips and Kim (2007). In Theorem 1, they show that the autocovariance matrix $\Gamma_{XX}(\tau)$ of a q-dimensional multivariate fractionally integrated process X_t is

$$[\Gamma_{XX}(\tau)]_{ab} = \frac{2f_{u_a u_b}(0)\Gamma(1 - d_a - d_b)\sin(\pi d_b)}{\tau^{1-d_a-d_b}} + O\left(\frac{1}{\tau^{2-d_a-d_b}}\right),$$

where A_{ab} denotes the element in the ath row and bth column of the matrix A. The asymptotic expansion of the Fourier integral used to derive this result is not specific to fractionally integrated processes, but holds for long memory processes in general. It therefore follows, that $\xi(\tau) = G_{xy}\tau^{d_a+d_b-1} + o(\tau^{d_a+d_b-1})$. Furthermore, since by Assumption 1 both x_t and y_t are causal, $\xi(\tau)\xi(-\tau) \to 0$, so that the last term in (1) drops out.

Therefore, as $\tau \to \infty$, we have

$$\gamma_{xy}(\tau) = \mu_x^2 G_y\tau^{2d_y-1} + \mu_y^2 G_x\tau^{2d_x-1} + \mu_x\mu_y G_{xy}\tau^{d_x+d_y-1} \\ + G_x G_y\tau^{2(d_x+d_y-1)} + o(\tau^{d_x+d_y-1}).$$

Now, considering the exponents and setting $d_x + d_y - 1 = 2\bar{d}_3 - 1$ and $2(d_x + d_y - 1) = 2\bar{d}_4 - 1$ gives $\bar{d}_3 = (d_x + d_y)/2$ and $\bar{d}_4 = (d_x + d_y - 1/2)$, so that

$$\gamma_{xy}(\tau) = \mu_x^2 G_y\tau^{2d_y-1} + \mu_y^2 G_x\tau^{2d_x-1} + \mu_x\mu_y G_{xy}\tau^{2\bar{d}_3-1} \\ + G_x G_y\tau^{2\bar{d}_4-1} + o(\tau^{d_x+d_y-1}). \quad (4)$$

Since $O(\tau^p) + O(\tau^q) = O(\tau^{\max(p,q)})$, the autocovariance function $\gamma_{xy}(\tau)$ is dominated by the term with the largest memory parameter, as $\tau \to \infty$. The approximation error $o(\tau^{d_x+d_y-1})$ vanishes, because $d_x, d_y < 1/2$. Depending on the values of μ_x and μ_y, different cases can be distinguished.

1. If $\mu_x = \mu_y = 0$, (4) is reduced to $\gamma_{xy}(\tau) \approx G_x G_y\tau^{2\bar{d}_4-1}$. Therefore, the memory of $x_t y_t$ would be given by $\bar{d}_4 = (d_x + d_y - 1/2)$, which can be negative so that the decay rate of the autocovariance function is that of an antipersistent LM process. However, in this case the long memory definition is only fulfilled if the spectral density is zero at the origin, which is equivalent to $S_{xy} = 0$. Otherwise the process is $LM(0)$.
2. If $\mu_x = 0 \neq \mu_y$, (4) becomes $\gamma_{xy}(\tau) \approx \mu_y^2 G_x\tau^{2d_x-1} + G_x G_y\tau^{2\bar{d}_4-1}$ and the dominating term is the maximum of d_x and $\bar{d}_4 = (d_x + d_y - 1/2)$. This is d_x, because $d_y < 1/2$.
3. If $\mu_y = 0 \neq \mu_x$, by the same arguments, the memory is d_y.
4. Finally, if $\mu_x, \mu_y \neq 0$, the memory order is the maximum of d_x, d_y, $(d_x + d_y)/2$ and $d_x + d_y - 1/2$. Furthermore, since $d_x, d_y < 1/2$, $\max\left\{d_x, d_y\right\}$ will always be at least as large as the other two terms.