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Coincidence of two solutions to Nash's bargaining problem

Todd Stambaugh

Department of Mathematics, CUNY Graduate Center, 365 Fifth Avenue, New York, NY 10016, United States

HIGHLIGHTS

- A summary is given for the Nash and Kalai–Smorodinsky solutions to the bargaining problem.
- An account is given for the precise conditions under which the two solutions coincide.
- In normalized situations, these solutions coincide only when they satisfy a number of notions of fairness.

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ABSTRACT

In 1950, John Nash gave an elegant solution to the bargaining problem using his somewhat controversial IIA axiom. Twenty-five years later, Ehud Kalai and Meir Smorodinsky gave a different solution replacing the IIA condition with their own Monotonicity condition. While the two solutions obviously coincide under certain conditions (e.g. when the problem is symmetric), they do not in general agree. This paper presents a complete account of the precise conditions under which the two solutions coincide.

1. Introduction

In interpersonal interactions, business negotiations, intergovernmental diplomacy, and in a wide variety of other contexts, notions of fairness play an essential role. In general, it is obviously quite difficult to determine what is a fair outcome to a, interaction. Even in situations where calculating the benefits gained for each side in an interaction is reasonably easy, the answer to what fairness means is still not necessarily clear.

In his 1950 paper, *The Bargaining Problem* (Nash, 1950), John Nash gave an axiomatic account of fairness in such scenarios. According to this approach, for any bargaining situation, one could determine a single fair bargain (or set of equivalent fair bargains). While the axioms for this solution seem reasonable, one in particular, the independence of irrelevant alternatives, came under some criticism. Partly as a result of this criticism, in 1975 Ehud Kalai and Meir Smorodinsky presented a new solution to the problem which avoided the IIA axiom entirely (Kalai and Smorodinsky, 1975).

These two solutions will yield the same bargain in certain situations, and different bargains in others. Clearly the two accounts of fairness differ not only philosophically, but practically as well. Determining when the two will agree and disagree can help us differentiate between those situations in which what is fair is in a sense obvious and those in which fairness is harder to pin down. This in turn can illuminate the practical implications of differing notions of fairness and whether our intuitions about fairness are reflected in the given axioms.

2. Formulation of the problem

A two-person bargaining situation can be represented as a pair (a, S), where S is a subset of $\mathbb{R}^2$ and $a = (a_1, a_2)$ is a point in S called the base point or disagreement point. Every point $x = (x_1, x_2) \in S$ represents the utilities, to agents 1 and 2 respectively, of reaching a particular bargain, with the base point representing the utilities received if the negotiations are abandoned. Every pair (a, S) must satisfy the following properties:

$\exists x \in S$ such that $x_i > a_i$ for $i = 1, 2$. (Bargaining Incentive)

S is convex. (Convexity)

S is compact. (Compactness)

$\forall x \in S, a_i \leq x_i,$ for $i = 1, 2$. (Individual Rationality)

Bargaining Incentive reflects the natural assumption that both agents would only engage in a bargaining situation in which they each stand to gain. If $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are the utilities

E-mail addresses: tstambaugh@gradcenter.cuny.edu, toddst11@gmail.com.

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associated to two potential bargains, any probability combination of the two bargains would itself be a bargain, yielding utilities $px + (1-p)y$ for some $p \in [0, 1]$, giving the convexity of S. Compactness in this context means that S is closed and bounded, which is reasonable from the realistic features of many situations to which this work would apply. Individual Rationality reflects the idea that no player would agree to a bargain in which he is worse off than he would be not bargaining at all. Individual Rationality is not assumed by Nash, though all of his results still hold with this included, as it only limits the space of possible bargaining pairs. Again this is a natural requirement, as no player would knowingly agree to a deal which gives a worse utility than disagreement.

Let $\mathcal{U}$ be the set of all bargaining pairs (a, S) with the above properties. A solution to the bargaining problem is a function $f : \mathcal{U} \to \mathbb{R}^2$ such that $f(a, S) = (f_1(a, S), f_2(a, S)) \in S$. In general, f is meant to give the "fair" agreement for the two agents.

3. Normalized bargaining pairs

We will only consider solutions that are invariant under affine transformations of utility, i.e. transformations of the form $A(x_1, x_2) = (c_1x_1 + d_1, c_2x_2 + d_2)$ for $c_1, c_2, d_1, d_2 \in \mathbb{R}$.[2] This means first that regardless the original base point a, we may consider without loss of generality the bargaining pairs $(0, S)$ (by translating the original base point). Further, let

$$b_1(S) = \sup\left\{x_1 \in \mathbb{R} \,\middle|\, \text{there exists } x_2 \in \mathbb{R}, (x_1, x_2) \in S\right\}$$
$$b_2(S) = \sup\left\{x_2 \in \mathbb{R} \,\middle|\, \text{there exists } x_1 \in \mathbb{R}, (x_1, x_2) \in S\right\}$$
$$b(S) = (b_1(S), b_2(S)).$$

As before, we may consider only bargaining pairs $(0, S)$ such that $b(S) = (1, 1)$ (by scaling, we can still keep the base point $(0, 0)$).

A bargaining pair is called *normalized* if $a = (0, 0)$ and $b(S) = (1, 1)$.

4. Nash solution η

Nash gives three axioms regarding properties that a solution should satisfy, along with philosophical justifications.

- N1 Pareto Efficiency[3]: For every bargaining pair (a, S), if $x \in S$ such that $\exists y \in S$ with $y_1 > x_1$ and $y_2 > x_2$, then $x \neq f(a, S)$.
- N2 Symmetry: If S is symmetric with respect to the line $x_1 = x_2$, then $f(0, S)$ lies on the line $x_1 = x_2$.
- N3 Independence of Irrelevant Alternatives (IIA): If (a, S) and (a, T) are bargaining pairs such that $S \subset T$ and $f(a, T) \in S$, then $f(a, S) = f(a, T)$.

N1 reflects the assumption that each agent is interested in maximizing her own utility. We assume that the players are equally skilled at negotiating, thus N2. N3 has fallen under serious criticism, including by Kalai and Smorodinsky (1975), but according to the intuition given by Nash, if two rational individuals would agree that $f(T)$ is fair if T were the set of possible bargains, then they should be willing to agree to the same deal with a smaller set $S \subset T$ of bargains available to them.

Nash proved that there is a unique function, η, given below, satisfying these three axioms.

$$\eta(a, S) = (\eta_1, \eta_2) \text{ where } (\eta_1, \eta_2) \in S \text{ and}$$
$$(\eta_1 - a_1)(\eta_2 - a_2) \geq (x_1 - a_1)(x_2 - a_2) \text{ for any } x \in S.$$

[2] Nash assumes this as a property of the utilities defined for each player. Kalai and Smorodinsky give this as an axiom for a solution. Significant philosophical objections have been brought up regarding this assumption including by Rubinstein et al. (1992), but for the purposes that follow, we may ignore these issues.

[3] Kalai and Smorodinsky use the term "Pareto Optimality" for this condition.

5. Kalai–Smorodinsky solution μ

Kalai and Smorodinsky give a different set of axioms that a solution must satisfy, motivated by issues raised regarding N3. Before proceeding we introduce some additional notation.

Let $g_S(x_1)$
$$= \sup\left\{x_2 \in \mathbb{R} \,\middle|\, x_1 \leq x_1' \text{ and } (x_1', x_2) \in S\right\} \text{ (defined for } x_1 \leq b_1).$$

Intuitively, $g_S(x_1)$ is the greatest utility the agent 2 can get if agent 1 gets at least x_1. A similar function can be defined for agent 1, but with symmetry, this is not necessary.

- KS1 Pareto Efficiency: For every bargaining pair (a, S), if $x \in S$ such that $\exists y \in S$ with $y_1 > x_1$ and $y_2 > x_2$, then $x \neq f(a, S)$.
- KS2 Symmetry: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T((x_1, x_2)) = (x_2, x_1)$. For every bargaining pair (a, S), $f(T(a), T(S)) = T(f(a, S))$.
- KS3 Invariance with Respect to Affine Transformations of Utility: If $A : \mathbb{R}^2 \to \mathbb{R}^2$, $A(x_1, x_2) = (c_1x_1 + d_1, c_2x_2 + d_2)$ for constants c_1, d_1, c_2, and d_2, then $f(A(a), A(s)) = A(f(a, S))$.
- KS4 Monotonicity: If (a, S) and (a, T) are bargaining pairs such that $b_1(S) = b_1(T)$ and $g_S \leq g_T$, then $f_2(a, S) \leq f_2(a, T)$.

The Pareto axiom remains unchanged. The purpose of the Symmetry axiom is the same, though it is formulated differently. This statement implies the Nash's, though both solutions satisfy both versions of the axiom. KS4 reflects the idea that if for any demand agent 1 can make, the maximum possible utility of agent 2 increases, then the utility for agent 2 under the solution should not decrease. KS3 reflects an assumption about the nature of the utility functions which define S, namely that they are determined up to changes in scale.

Kalai and Smorodinsky also give a unique function μ which satisfies this new set of axioms.

$\mu(a, S) = (\mu_1, \mu_2)$
is the maximal point in S on the line through a and $b(S)$.

6. Coincidence of η and μ

Kalai and Smorodinsky showed by example that η does not satisfy KS4, so in general $\eta \neq \mu$ (Kalai and Smorodinsky, 1975). However, it is clear from the Symmetry axioms for both solutions that the two will always coincide when S is symmetric about the line $x_1 = x_2$. It is easy to build an example for which S is not symmetric, but the two solutions are the same, meaning symmetry alone will not predict coincidence.[4]

Example. Let S' be the convex hull of the points $(0, 0)$, $(1, 0)$, $(0, 1)$, $(0.9, 0.9)$, $(0.5, 1)$, and $(1, 0.5)$. It is easy to check that $\eta(0, S') = \mu(0, S') = (0.9, 0.9)$

Let S be the convex hull of $(0, 0)$, $(1, 0)$, $(0, 1)$, $(0.9, 0.9)$ and $(0.5, 1)$ which is clearly not symmetric. $S \subset S'$ both of which contain $\eta(0, S')$, so $\eta(0, S) = \eta(0, S')$. Since $b(S) = b(S') = (1, 1)$ and the line segment between $(0, 0)$ and $(0.9, 0.9)$ is contained in S, $\eta(0, S) = \eta(0, S') = \mu(0, S)$.

It turns out that there is a simple characterization of the properties of (a, S) for which $\eta(a, S) = \mu(a, S)$.[5]

To approach this question, it is sensible to first restrict our attention to normalized bargaining pairs for the sake of simplicity.

[4] Credit is due to Rann Smorodinsky for giving an example of such a bargaining pair, different from the example included.

[5] This question was brought up by Rohit Parikh of the CUNY Graduate Center, who was instrumental in the completion of this work.