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Fixed bandwidth asymptotics for the studentized mean of fractionally integrated processes[☆]

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HIGHLIGHTS

- We consider inference for the mean of a general stationary process.
- We use a frequency domain estimator of the long run variance.
- We consider alternative asymptotics in which the bandwidth is kept fixed.
- The fixed-bandwidth limit appears to be more precise than traditional asymptotics.

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ABSTRACT

We consider inference for the mean of a general stationary process based on standardizing the sample mean by a frequency domain estimator of the long run variance. Here, the main novelty is that we consider alternative asymptotics in which the bandwidth is kept fixed. This does not yield a consistent estimator of the long run variance, but, for the weakly dependent case, the studentized sample mean has a Student-*t* limit distribution, which, for any given bandwidth, appears to be more precise than the traditional Gaussian limit. When data are fractionally integrated, the fixed bandwidth limit distribution of the studentized mean is not standard, and we derive critical values for various bandwidths. By a Monte Carlo experiment of finite sample performance we find that this asymptotic result provides a better approximation than other proposals like the test statistic based on the Memory Autocorrelation Consistent (MAC) estimator of the variance of the sample mean.

1. Introduction

We consider inference for the mean of a covariance stationary time series with autocorrelation of unspecified nature. This problem has been widely studied for weakly dependent processes (whose spectral density is finite and nonzero), where robust inference can be obtained by standardizing the sample average by the long run variance: this is usually unknown, but it can be estimated by a range of techniques based on either weighted autocovariances or weighted periodograms, see, e.g., Priestley (1981).

In the present paper we emphasize the use of frequency domain techniques. In this setting, the simplest estimator of the long run variance can be obtained by direct averaging periodograms evaluated at the first m Fourier frequencies (which corresponds to using the Daniell kernel), where m is known as bandwidth. When discussing the limiting properties of this estimator, it is routinely assumed that $m \to \infty$, although at a rate slower than the sample size T, so that the band m/T is degenerating to 0. Throughout, we will denote the assumption $m \to \infty$ as the large-m approach. Here, we will consider instead an alternative strategy to derive the asymptotic properties in which m is kept fixed. The motivation is that in any practical situation a finite m is used, so letting the asymptotic distribution depend on a fixed m might yield a better approximation to the sampling distribution of the corresponding test statistic. This approach will be denoted as fixed-m, and can be seen as a frequency domain analogue to the fixed-b approach for time domain estimators of the long run variance. The fixed-b strategy has been also applied to provide a more accurate limit approximation to the sampling distribution of the studentized

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mean (see, e.g., Kiefer and Vogelsang, 2002, 2005, Jansson, 2004, Sun et al., 2008, McElroy and Politis, 2012, 2013).

Despite the analogies, the fixed-m and fixed-b limits are different and, as we show in the paper, in the leading case of weakly dependent processes, the fixed-m approach does not require the simulation of a null limit distribution (which is found to be a Student-t with $2m$ degrees of freedom, t_{2m}). However, weak dependence is just a particular case of the general type of dependence we allow for in the paper, which is captured by the so-called fractional processes. This includes the long memory and antipersistent situations, with positive and negative memories, respectively, where the fixed-m limit of the standardized mean is not standard and we derive critical values for various bandwidths.

The following section presents the studentized mean and discusses its large-m and fixed-m limits. In Section 3, we compare the large-m and fixed-m limiting approximations to the sampling distribution of the studentized mean by a Monte Carlo experiment. Finally, in Section 4 we conclude. Proofs are given in the Appendix.

2. Large-m and fixed-m limits of the studentized mean

We consider the time series $x_1, \ldots, x_T$, observed from the stationary process $x_t := \mu + u_t$, where $E(u_t) = 0$ and u_t may be subject to a general type of dependence characterized in Assumptions 1 and 2 below.

Assumption 1. Let $\eta_t = A(L)\varepsilon_t := \sum_{j=0}^{\infty} A_j \varepsilon_{t-j}$ where L is the usual lag operator. The weights $\{A_j\}$ are such that $A(1)^2 > 0$ and $\sum_{l=0}^{\infty} l|A_l| < \infty$ and ε_t is an independent, identically distributed (i.i.d.) sequence with $E(\varepsilon_t) = 0, E\left(\varepsilon_t^2\right) = 1$.

Assumption 2. Let $\Delta_t^{(\delta)} := \Gamma(t+\delta)/(\Gamma(\delta)\Gamma(t+1))$, $\Gamma(\cdot)$ denoting the Gamma function, such that $\Gamma(0) := \infty$ and $\Gamma(0)/\Gamma(0) := 1$, and $u_t := \sum_{s=-\infty}^{t} \Delta_{t-s}^{(\delta)} \eta_s$, $\delta \in (-1/2, 1/2)$.

Assumptions 1, 2 imply that, in general, u_t is a Type 1 fractionally integrated process. We consider inference on μ when the dependence structure of u_t is not known. In this case, the sample mean $\bar{x} := T^{-1}\sum_{t=1}^{T} x_t$, is a natural estimator of μ and, if u_t is weakly dependent, that is $u_t = \eta_t$ (or $\delta = 0$) and regularity conditions are met, inference on $\bar{x}$ can be based on the Central Limit Theorem (CLT)

$$\sqrt{T}(\bar{x}-\mu)/\sigma \to_d N(0,1), \tag{1}$$

where $\sigma^2 := A(1)^2$ is typically known as long run variance. In practice σ^2 is unknown, but a large number of semiparametric techniques are available to estimate it consistently, see, e.g., Priestley (1981). Letting $w_x(\lambda) := (2\pi T)^{-1/2}\sum_{t=1}^{T} x_t e^{i\lambda t}$ be the Fourier transform of x_t and the periodogram $I(\lambda) := |w_x(\lambda)|^2$, the Daniell kernel provides a very simple estimator of σ^2, $\widehat{\sigma}^2 := 2\pi \frac{1}{m}\sum_{j=1}^{m} I\left(\lambda_j\right)$, where $\lambda_j := 2\pi j/T$. Feasible inference is then conducted using the statistic

$$\tau := \sqrt{T}(\bar{x}-\mu)/\widehat{\sigma}. \tag{2}$$

When $m \to \infty$, $m/T \to 0$ and given other regularity conditions, $\widehat{\sigma}^2$ is consistent, and it can be substituted in (1) without altering the limit. To derive the fixed-m asymptotic distribution of $\widehat{\sigma}^2$, where m is kept fixed, we need to strengthen the moment conditions on ε_t.

Assumption 3. There is q such that $E\left(|\varepsilon_t|^q\right) < \infty$ with $q \geq \max(2, 2/(1+2\delta))$.

Remark 1. Under Assumptions 1–3, the following Functional Central Limit Theorem (FCLT) for fractional process holds: for $r \in [0,1]$, as $T \to \infty$,

$$\frac{1}{T^{1/2+\delta}}\sum_{t=1}^{\lfloor rT\rfloor} u_t \Rightarrow \Sigma_\delta W_{\delta+1}(r), \tag{3}$$

where $W_{\delta+1}(r)$ is a Type I fractional Brownian motion, as defined in Mandelbrot and Van Ness (1968),

$\Sigma_\delta^2 := \sigma^2\Gamma(1-2\delta)/[(1+2\delta)\Gamma(1+\delta)\Gamma(1-\delta)]$

and $\lfloor\cdot\rfloor$ denotes integer part. For further details see Theorems 2.1 and 2.2 of Wang et al. (2003). When $\delta = 0$ the limit (3) encompasses the standard convergence to the standard Brownian motion. Let $\widehat{W}_{\delta+1}(r) := W_{\delta+1}(r) - rW_{\delta+1}(1)$ and

$$Q_\delta(j) := \left\{\left(2\pi j\int_0^1 \sin(2\pi jr)\widehat{W}_{\delta+1}(r)\,dr\right)^2 + \left(2\pi j\int_0^1 \cos(2\pi jr)\widehat{W}_{\delta+1}(r)\,dr\right)^2\right\}.$$

Our key result is Lemma 1 below.

Lemma 1. *Under Assumptions 1–3, for fixed $j = 1, \ldots, m$, as $T \to \infty$,*

$$T^{-2\delta}2\pi I\left(\lambda_j\right) \to_d \Sigma_\delta^2 Q_\delta(j). \tag{4}$$

By Lemma 1 and (3) we can establish the following theorem.

Theorem 2. *Under Assumptions 1–3, for fixed m, as $T \to \infty$,*

$$\tau \to_d \frac{W_{\delta+1}(1)}{\sqrt{\frac{1}{m}\sum_{j=1}^{m} Q_\delta(j)}}. \tag{5}$$

When $\delta = 0$, it is well known that, under regularity conditions, the joint distribution of $2\pi I\left(\lambda_j\right)$, $j = 1, \ldots, m$, converges to that of m independent $2^{-1}\sigma^2\chi_2^2$ variates (see, e.g., Theorem 13 and pp. 225, 226 of Hannan, 1970). Then, by the continuous mapping theorem and exploiting also the asymptotic independence of $I(0)$ and $I\left(\lambda_j\right)$, $j = 1, \ldots, m$, it is straightforward to derive.

Corollary 3. *Under Assumptions 1 and 2 and $\delta = 0$, for fixed m, as $T \to \infty$,*

$\tau \to_d t_{2m}.$

Remark 2. In related settings, the Student-t limiting distribution has already been posed by Sun (2013) (Theorem 3.1) and Müller (2014). In particular, our Corollary 3 justifies formally the heuristic discussion of Müller (2014, p. 314), who anticipated that, under weak dependence, taking into account the uncertainty in $\widehat{\sigma}^2$ instead of relying on consistency arguments, leads to a Student-t limiting result instead of the traditional $N(0,1)$ limit. Thus our Theorem 2, which generalizes this result allowing also for long memory and antipersistency, encompasses Müller's (2014) claim.

Remark 3. Asymptotic expected values and correlation of periodograms are derived in Hurvich and Beltrao (1993), where a limit distribution of the periodogram for the Gaussian case is also given. Asymptotics for the Fourier transforms of possibly fractionally integrated processes are also given in Chen and Hurvich (2003) and