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A note on the envelope theorem

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ABSTRACT

The purpose of this note is to discuss the envelope relationship between long run and short run cost functions. It compares the usually presented relationship with one of different form and implications, resulting from a simple production function and constant prices. It points out in particular that the tangency condition between the short and long run total cost functions does not necessarily hold always. The note also shows that a given value of the fixed factor might support in the long run a whole range of levels of output.

1. Introduction

We wish to consider the relation between the short run and the long run cost functions in the context of two examples, in which the factors of production can assume any non-negative values. The first example gives rise to the usual textbook diagram while the second one does not. It is precisely the possibility of the second example that is the reason for this note. A main implication of the analysis is that the tangency condition between the short and long run total cost functions does not necessarily hold always. Also there is a given, optimal value of the fixed factor which in the long run will support all outputs beyond a particular level. Of course the quantity of the variable factor will adjust itself.

In the economics literature there have been some discussions of applications of a generalized envelope theorem (see for example Benveniste and Scheinkman, 1979, Milgrom and Segal, 2002, Mas-Colell et al., 1995). On the other hand, in general in economics, and in particular in advanced textbooks, the envelope property is discussed in the context of equality between the tangents of short run and long run cost functions, (see Luenberger, 1995, Mankiw, 2001, Pindyck and Rubinfeld, 2005, Simon and Blume, 1994, Varian, 1992, 2003). Here we engage in a generalization of the envelope theorem where the possibility of a corner solution is also present.

2. Examples of the envelope theorem

We discuss the following two examples. The short run and long run total cost functions are denoted respectively by C_S^* and C_L^*.

Example 1. We consider the simple model $Y = x_1^{\alpha} x_2^{\beta}$ where $\alpha, \beta > 0$ and $\alpha + \beta = 1$, and $x_1 \in \mathbb{R}_{\geq 0}$ the variable and $x_2 \in \mathbb{R}_{\geq 0}$ the fixed inputs in the short run. For the prices we assume $p_1, p_2 > 0$. We show below that the relation between the cost functions is the conventional one.

The short run

Given the value of x_2 we obtain the demand function $x_1 = \left(\frac{Y}{x_2^{\beta}}\right)^{1/\alpha}$, and the short run cost function, $C_S^* = p_1 \frac{Y^{1/\alpha}}{x_2^{\beta/\alpha}} + p_2 x_2$ which is rising and convex in Y.

The short run average and marginal cost functions are, respectively, $A_S^* = p_1 \frac{Y^{\beta/\alpha}}{x_2^{\beta/\alpha}} + p_2 \frac{x_2}{Y}$ and $M_S^* = p_1 \frac{1}{\alpha} \frac{Y^{\beta/\alpha}}{x_2^{\beta/\alpha}}$.

The long run

In order to obtain the long run cost function, C_L^*, where x_2 is also allowed to vary continuously, we can minimize C_S^* with respect to x_2. We have the first order condition $\frac{dC_S^*}{dx_2} = -p_1 \frac{\beta}{\alpha} \frac{Y^{1/\alpha}}{x_2^{(\beta/\alpha)+1}} + p_2 = 0$, and second order condition, $\frac{dC_S^{*2}}{dx_2^2} > 0$.

Solving the first order condition we obtain $x_2 = \left(\frac{p_1}{\alpha}\frac{\beta}{p_2}\right)^{\alpha} Y$ and substituting into C_S^* we obtain the long run cost function $C_S^{**} = C_L^*$

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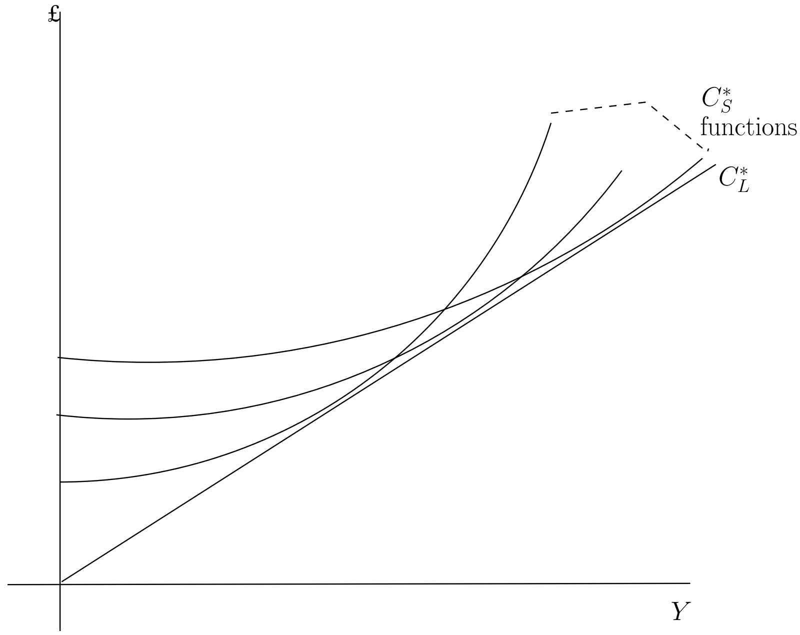

Fig. 1. The cost functions for $Y = x_1^{\alpha} x_2^{\beta}$ with $\alpha, \beta > 0, \alpha + \beta = 1$ and $p_1, p_2 = 1$.

$= \left[p_1 \left(\frac{\alpha}{p_1} \frac{p_2}{\beta} \right)^{\beta} + p_2 \left(\frac{p_1}{\alpha} \frac{\beta}{p_2} \right)^{\alpha} \right] Y = \left(\frac{p_1}{\alpha} \right)^{\alpha} \left(\frac{p_2}{\beta} \right)^{\beta} Y$, where $C_S^{**} =$ minimumC_S^*.

The function C_L^* can also be obtained from the cost minimization problem:

Minimize $C = p_1 x_1 + p_2 x_2$

Subject to

$$Y = x_1^{\alpha} x_2^{\beta}, \quad x_1, \ x_2 \geq 0,$$

where $p_1, \ p_2, \ Y$ are fixed.

It is easy to see that the long run demand functions of the inputs are $x_1 = \left(\frac{\alpha}{p_1} \frac{p_2}{\beta} \right)^{\beta} Y$ and $x_2 = \left(\frac{p_1}{\alpha} \frac{\beta}{p_2} \right)^{\alpha} Y$, where the expression for x_2 is identical to the one that results from the condition $\frac{dC_S^*}{dx_2} = 0$. These demand functions imply, of course, the expression of C_L^* obtained above.

The long run average and marginal cost functions are:

$$A_L^* = M_L^* = \left[p_1 \left(\frac{\alpha}{p_1} \frac{p_2}{\beta} \right)^{\beta} + p_2 \left(\frac{p_1}{\alpha} \frac{\beta}{p_2} \right)^{\alpha} \right] = \left(\frac{p_1}{\alpha} \right)^{\alpha} \left(\frac{p_2}{\beta} \right)^{\beta}.$$

C_L^* is the envelope of the C_S^* curves and A_L^* that of the A_S^* ones. In both cases every point of the envelope curve corresponds to a point of a unique short run curve. This is the usual case when the fixed factor of production can vary continuously.

The tangency condition between the minimum of the C_S^* convex curves, given Y, and C_L^* follows from the fact that all functions are smooth and C_L^* is obtained from a minimization problem with an interior solution. This is looked at again in the Appendix.

The connection between C_S^* and C_L^* is shown diagrammatically in Fig. 1, where without loss of generality we have taken[1] $p_1, \ p_2 = 1$. The resulting relation between A_S^* and A_L^* is shown[2] in Fig. 2. At the point of equality of the total cost curves we also have $M_S^* = M_L^*$. This follows from the fact that the marginal cost is the derivative of the total cost, and from the tangency condition between the C_S^* and the C_L^* curves. This equality holds precisely for that level of output. The tangency of the total curves implies the tangency of the average functions. We return to this in the Appendix.

Now we wish to investigate the shape of the A_S^* curve. The first and second order derivatives of A_S^* with respect to Y are

$$\frac{dA_S^*}{dY} = p_1 \frac{\beta}{\alpha} \frac{Y^{(\beta/\alpha)-1}}{x_2^{\beta/\alpha}} - p_2 \frac{x_2}{Y^2}$$

and $$\frac{d^2 A_S^*}{dY^2} = p_1 \frac{\beta}{\alpha} \left(\frac{\beta}{\alpha} - 1 \right) \frac{Y^{(\beta/\alpha)-2}}{x_2^{\beta/\alpha}} + p_2 \frac{2x_2}{Y^3}.$$

The sign of $\frac{d^2 A_S^*}{dY^2}$ is the same as that of $p_1 \frac{\beta}{\alpha} \left(\frac{\beta}{\alpha} - 1 \right) \frac{Y^{\beta/\alpha}}{x_2^{\beta/\alpha}} + p_2 \frac{2x_2}{Y}$. It follows that for $\frac{\beta}{\alpha} - 1 \geq 0$ the A_S^* curve is convex throughout.

Next, we wish to investigate the case $\frac{\beta}{\alpha} - 1 < 0$. First we see what happens around the point of tangency of A_S^* with A_L^*. At that point we have $\frac{dA_S^*}{dY} = \frac{dA_L^*}{dY} = 0$ which implies $\frac{Y^{\alpha/\beta}}{x_2^{\alpha/\beta}} = \frac{\alpha}{\beta} \frac{p_2}{p_1} \frac{x_2}{Y}$, and substituting into the expression $p_1 \frac{\beta}{\alpha} \left(\frac{\beta}{\alpha} - 1 \right) \frac{Y^{\beta/\alpha}}{x_2^{\beta/\alpha}} + p_2 \frac{2x_2}{Y}$ we get the equality $p_1 \frac{\beta}{\alpha} \left(\frac{\beta}{\alpha} - 1 \right) \frac{\alpha}{\beta} \frac{p_2}{p_1} \frac{x_2}{Y} + p_2 \frac{2x_2}{Y} = \left(\frac{\beta}{\alpha} - 1 \right) p_2 \frac{x_2}{Y} + p_2 \frac{2x_2}{Y}$; this is of the same sign as $\frac{\beta}{\alpha} + 1$ which is positive.

Therefore at the point of tangency with A_L^* the function A_S^* is convex.

Next we look at the total behaviour of the function $\frac{d^2 A_S^*}{dY^2}$. As noted above, its sign is determined by that of the expression $p_1 \frac{\beta}{\alpha} \left(\frac{\beta}{\alpha} - 1 \right) \frac{Y^{\beta/\alpha}}{x_2^{\beta/\alpha}} + p_2 \frac{2x_2}{Y}$. Due to the fact that $\left(\frac{\beta}{\alpha} - 1 \right) < 0$, for sufficient large Y it turns and stays concave. The concave part is beyond the point of tangency and it is of course rising to the right and falling to the left of this point.

Example 2. The production function is now given by $Y = x_1 + 2x_2^{0.5}$, where, the non-negative, x_1 is the variable and x_2 the fixed inputs in the short run. The isoquants correspond to fixed Y and they have slope $dx_1 + x_2^{-0.5} dx_2 = 0$. They are shown in Fig. 3.

[1] All figures are drawn under the assumption that $p_1 = p_2 = 1$.

[2] We note that for $\alpha + \beta > (<) 1$, i.e. for the case of increasing (decreasing) returns to scale, the C_L^* curve will be concave (convex), and the A_L^* one will be decreasing (increasing). Also, in the case of increasing returns to scale the declining A_L^* curve will be convex.