Economics Letters 149 (2016) 102–107

Contents lists available at ScienceDirect

Economics Letters

journal homepage: www.elsevier.com/locate/ecolet

Heavy tails and copulas: Limits of diversification revisited☆

Rustam Ibragimov [a,b], Artem Prokhorov [c,d,*]

[a] Imperial College Business School, UK
[b] Innopolis University, Russia
[c] The University of Sydney Business School, Australia
[d] St.Petersburg State University, Russia

ARTICLE INFO

Article history:
Received 27 September 2016
Accepted 19 October 2016
Available online 26 October 2016

JEL classification:
C58
C13
G11

Keywords:
Value at risk
Diversification
Power law
Power-type copulas

ABSTRACT

We show that diversification does not reduce Value-at-Risk for a large class of dependent heavy tailed risks. The class is characterized by power law marginals with tail exponent no greater than one and by a general dependence structure which includes some of the most commonly used copulas.

1. Introduction

Level-q Value-at-Risk VaR_q ($q > 0$), also known as the level-q quantile of a distribution of losses, is a commonly used risk measure, whose popularity in a wide range of areas in finance is attributed to the recommendations of the Basel Committee on Banking Supervision. A series of recent papers studied the problem of portfolio optimization in the VaR framework, mostly focusing on the situation when the portfolio components are independent and have a heavy tailed distribution (see, e.g., Embrechts et al., 1997, 2009b; Ibragimov and Walden, 2011). An important conclusion from that work is that if the tails of return distributions are extremely heavy then diversification increases portfolio riskiness in terms of VaR.

This property of VaR known as non-subadditivity has been studied in i.i.d. settings by many authors. For example, Garcia et al. (2007), Ibragimov and Walden (2007) and Ibragimov (2009b) focus on i.i.d. stable random variables (r.v.'s) with infinite variance and show that VaR is subadditive provided that the mean of the distribution is finite. Similar results are obtained for asymptotically large losses of portfolios of i.i.d. risks with general power law distributions.

There are a few extensions to non-independence. For example, Ibragimov and Walden (2007, 2011) consider dependence arising from common multiplicative and additive shocks, Embrechts et al. (2009b) and Chen et al. (2012) consider Archimedian copulas, Asmussen and Rojas-Nandayapa (2008) consider the normal copula, Albrecher et al. (2006) consider Archimedian copulas. Barbe et al. (2006) use a spectral measure of the tail dependence. Embrechts et al. (2009a) and Jessen and Mikosch (2006) use multivariate regular variation. The studies find that the subadditivity property of VaR is generally affected by both the strength of dependence and the tail behavior of the marginals, however in some cases only heavy tails of the marginals matter (see Ibragimov and Prokhorov, 2017, for a survey of these and related results).

The purpose of this paper is to provide new results on subadditivity of VaR in non-iid settings.

☆ Helpful comments from participants of CFE2014, CFE2015, NASMES2016, EMS2015, as well as seminar participants at Imperial College London, University of Queensland and St. Louis University are gratefully acknowledged. We thank James Diaz, William Liu, Jingyuan Mo and Fujie Xia for excellent research assistance. Research for this paper was supported by a grant from the Russian Science Foundation (Project No. 16-18-10432).

* Corresponding author at: The University of Sydney Business School, Australia.
E-mail addresses: irustam@imperial.ac.uk (R. Ibragimov), artem.prokhorov@sydney.edu.au (A. Prokhorov).

http://dx.doi.org/10.1016/j.econlet.2016.10.024
0165-1765/

2. Diversification under independence

2.1. Heavy tails and power law family

A tail index of a univariate distribution characterizes the heaviness, or the rate of decay, in the tails of the distribution, assuming it obeys a power law. Let X denote a loss. Then, X belongs to the *power law family* of distributions if

$$\mathbb{P}(|X| > x) \sim x^{-\alpha}, \tag{1}$$

where α is the tail index, or tail exponent, and "$\sim$" means that the left hand side is asymptotically equivalent to a nonzero constant times the right hand side (and asymptotics is with respect to $x \to \infty$). The power law family is commonly used in financial econometrics to model heavy tailed distributions (see, e.g., Gabaix, 2009; Ibragimov, 2009b).

Power law distributions are attractive because they permit modeling rates of tail decay that are slower than the exponential decay of a Gaussian distribution. Such distributions often form the basis of a wider class obtained by introducing a slight disturbance to the tail behavior in the form of a slowly varying function (see, e.g., Embrechts et al., 1997; Ibragimov and Walden, 2008). Many distributions can be viewed as special cases of power laws, at least for asymptotically large losses. This includes Pareto and Student-t distributions as well as Cauchy, Levy and other stable distributions with the index of stability $\alpha < 2$.

The tail index α governs the likelihood of observing outliers or large fluctuations of risks or returns in consideration: a smaller tail index means slower rate of decay of tails of risk distributions, which means that the above likelihood is higher. When the tail index is less than two, the tail decay is so slow that the second moment of the underlying risk or return distribution is infinite; when the tail index is less than one, the first moment is infinite. More generally, the power law distributions have the property that absolute moments of X are finite if and only if their order is less than tail index α. That is,

$$\mathbb{E}|X|^p < \infty \quad \text{if } p < \alpha; \qquad \mathbb{E}|X|^p = \infty \quad \text{if } p \geq \alpha.$$

A large number of studies in economics, finance and insurance have documented that financial returns and other important financial and economic variables have heavy-tailed distributions with values of α ranging from significantly lower than one to above four (Jansen and Vries, 1991; Loretan and Phillips, 1994; McCulloch, 1997; Rachev and Mittnik, 2000; Gabaix et al., 2006; Chavez-Demoulin et al., 2006; Silverberg and Verspagen, 2007, and references therein).

We will say that a risk has *extremely* heavy tails if $\alpha < 1$, and *moderately* heavy tails if $\alpha > 1$.

2.2. Limits of diversification under heavy tails and independence

Consider a simple problem of optimal portfolio allocation in the VaR framework with possibly extremely heavy tailed losses $X_j > 0$, $j = 1, 2$. Let $w = (w_1, w_2) \in \mathbb{R}^2$ be the portfolio weights such that $w_1, w_2 \geq 0$, $w_1 + w_2 = 1$. Consider the tail of the aggregate loss distribution $\mathbb{P}(w_1X_1 + w_2X_2 > x)$, where the weighted average loss $w_1X_1 + w_2X_2$ corresponds to a portfolio with weights w_1 and w_2. Unless one of the weights is zero, the portfolio is diversified.

A q% Value-at-Risk of a portfolio risk Z is $\text{VaR}_q(z) = \inf\{z \in \mathbb{R} : \mathbb{P}(Z > z) \leq q\}$, or the $(1 - q)$th quantile of the loss distribution. The problem of interest is to minimize $\text{VaR}_q(w_1X_1 + w_2X_2)$ over the weights w for a given $q \in (0, 1/2)$.

When X_1 and X_2 are i.i.d. with a stable distribution, it is now well understood that, for all non-zero w's, $\mathbb{P}(w_1X_1 + w_2X_2 > x) \leq \mathbb{P}(X_1 > x)$ if $\alpha_j > 1, j = 1, 2$. In other words, the VaR of a diversified portfolio of moderately (but not extremely) heavy tailed risks is no greater than that of a not diversified.

If X_j's are i.i.d. with $\alpha_j < 1$ then $\mathbb{P}(w_1X_1 + w_2X_2 > x) \geq \mathbb{P}(X_1 > x)$; that is, for extremely heavy-tailed risks the benefits of diversification disappear and the least risky portfolio is one that has a single risk. For example, if X_j's are i.i.d. stable with $\alpha = 1/2$, that is, if X_j's are Levy distributed, the aggregate loss of an equally weighted portfolio $\frac{X_1+X_2}{2}$ has the same distribution as $2X_1$ and thus $\text{VaR}_q\left(\frac{X_1+X_2}{2}\right) = 2\text{VaR}_q(X_1) > \text{VaR}_q(X_1)$.

Ibragimov (2009b) showed that analogous statements hold for portfolios of any size, using majorization theory. Similar results are available for bounded risks concentrated on a sufficiently large interval: for such cases, VaR-based diversification is suboptimal up to a certain number of risks and then becomes optimal (Ibragimov and Walden, 2007).

There is a growing range of applications of these seemingly counterintuitive results in finance, economics and insurance. Ibragimov et al. (2009) demonstrate how this analysis can be used to explain abnormally low levels of reinsurance among insurance providers in markets for catastrophic insurance. Ibragimov et al. (2011) show how to analyze the recent financial crisis as a case of excessive risk sharing between banks when risks are extremely heavy-tailed. Gabaix (2009) provides a review of applications of the above conclusions in different areas of economics and finance.

Let $(\xi_1(\alpha), \xi_2(\alpha))$ denote independent random variables from a power-law distribution with a common tail index α. It follows from the two-risks example above that the limits of diversification results hold for i.i.d. losses regardless of the weights w_j. In what follows we consider an equally weighted portfolio $w_1 = w_2 = 1/2$, for simplicity. All our results can be easily extended to a portfolio of size n with any weights, in which case the aggregate loss $\left(\frac{\xi_1(\alpha)+\xi_2(\alpha)}{2}\right)$ is replaced with $\sum_{i=1}^{n} w_i\xi_i(\alpha)$.

Theorem 1 (*Theorems 4.1 and 4.2 of Ibragimov, 2009b*). *For sufficiently small loss probability q,*

$$VaR_q\left(\frac{\xi_1(\alpha) + \xi_2(\alpha)}{2}\right) < VaR_q(\xi_1(\alpha)), \quad \text{if } \alpha > 1 \tag{2}$$

$$VaR_q\left(\frac{\xi_1(\alpha) + \xi_2(\alpha)}{2}\right) > VaR_q(\xi_1(\alpha)), \quad \text{if } \alpha < 1. \tag{3}$$

An interesting boundary case corresponds to $\alpha = 1$. This is when diversification has no effect, i.e. it neither increases nor reduces VaR. For example, if ξ's are i.i.d. stable with $\alpha = 1$, which means they have a Cauchy distribution, one has that $\frac{\xi_1(\alpha)+\xi_2(\alpha)}{2}$ has the same distribution as $\xi_i(\alpha)$, so a diversified and a non-diversified portfolios have identical VaRs.

It is not obvious what happens if we relax the independence assumptions. The two extreme cases, corresponding to a comonotone (extreme positive) and countermonotone (extreme negative) dependence do not present a consistent picture. For example, if we consider extreme positive dependence with $\xi_1 = \xi_2$ (a.s.) then, obviously, $\text{VaR}_q(w_1\xi_1(\alpha) + w_2\xi_2(\alpha)) = \text{VaR}_q(\xi_1(\alpha))$ and so diversification has no effect regardless of the tails; while if we have extreme negative dependence with $\xi_1 = -\xi_2$ (a.s.) then $\text{VaR}_q(w_1\xi_1(\alpha) + w_2\xi_2(\alpha)) = (w_1 - w_2)\text{VaR}_q(\xi(\alpha))$ and it is optimal to fully diversify regardless of the tails.

3. Diversification under dependence

3.1. Dependence and copulas

Copulas are joint distributions with uniform marginals. They are useful because given the marginal distributions, they represent the