Economics Letters 149 (2016) 164–167

Contents lists available at ScienceDirect

Economics Letters

journal homepage: www.elsevier.com/locate/ecolet

Network connectivity under node failure

Pascal Billand [a], Christophe Bravard [b], Sitharama S. Iyengar [c], Rajnish Kumar [d,*], Sudipta Sarangi [e,f]

[a] *Université de Lyon, St Etienne, France*
[b] *GAEL, Univ. Grenoble-Alpes, France*
[c] *Florida International University, FL, USA*
[d] *Queen's Management School, Queen's University Belfast, United Kingdom*
[e] *Virginia Tech, VA, USA*
[f] *DIW Berlin, Germany*

HIGHLIGHTS

- A model of network formation with the possibility of node failure has been analysed.
- Players receive benefits from connecting directly and indirectly through costly links.
- Conditions are identified for Nash and efficient networks.
- The conditions involve notion of *k*-Node Super Connectivity.

ARTICLE INFO

Article history:
Received 28 August 2016
Received in revised form
13 October 2016
Accepted 21 October 2016
Available online 26 October 2016

JEL classification:
C7
D8
R4

Keywords:
Connectivity
Node failure
Nash networks
Efficient networks

ABSTRACT

We examine a non-cooperative model of network formation where players may stop functioning. We identify conditions under which Nash and efficient networks will remain connected after the loss of k nodes by introducing the notion of k-Node Super Connectivity.

1. Introduction

In this paper, we focus on a model of network formation where players (also called nodes) in the network can fail with a certain exogenous probability. Examples of networks being affected by node failure abound in the real world. Consider for instance, the social and economic networks in a region hit by a natural calamity like a hurricane, or a business network where some firms exit the industry. This can also occur in a network of servers or a sensor network either due to mechanical failure or a malicious attack. Connectivity in the network is important in all these examples, suggesting that strategic agents will have an incentive to create multiple paths between themselves.

The literature on strategic reliability in economics has mostly focused on the possibility of links failures. The model incorporating reliability in networks was introduced in a paper by Bala and Goyal (2000) where all links are allowed to fail with a given exogenous probability. The authors then proceed to provide a partial characterization of Nash and efficient networks in this

* Correspondence to: Queen's Management School, Queen's University Belfast, 185 Stranmillis Road, Belfast, BT9 5EE United Kingdom. Fax: +44 (0)289097 4201.
E-mail addresses: pascal.billand@univ-st-etienne.fr (P. Billand), christophe.bravard@upmf-grenoble.fr (C. Bravard), iyengar@cis.fiu.edu (S.S. Iyengar), rajnish.kumar@qub.ac.uk (R. Kumar), ssarangi@vt.edu (S. Sarangi).

http://dx.doi.org/10.1016/j.econlet.2016.10.028

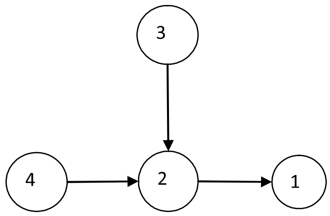

Fig. 1. Node failure vs. link failure: An illustration.

context. Subsequently, Haller and Sarangi (2005) and Billand et al. (2011) allow for heterogeneity in the link failure probabilities and values that can be obtained from other players respectively. While full characterization of the equilibrium networks is again shown to be elusive, both papers provide an "anything goes" result which shows that with only a bit of heterogeneity (two different parameter values) any essential network can be supported as a Nash equilibrium. In a slightly different framework, De Jaegher and Hoyer (forthcoming) study a game between a network designer and a network disruptor and find the equilibrium network architectures for different levels of link costs. These types of game have also been studied by Dziubiński and Goyal (2013), Goyal and Vigier (2014) and Haller (2016). Note that in these models the attacker typically removes links or nodes.

Ours is the first paper to study node failure when agents form links in a decentralized way—a phenomenon quite different from link failure. Under link failure, as well as under node failure, agents can create alternate paths by forming costly links with other agents. However, the logic for creating alternate paths is quite different. Consider Fig. 1.

In Fig. 1, player 1 has formed a link with player 2 and player 2 has formed links with players 3 and 4. In this figure, under link failure, the formation of an additional link by player 1 with player 3 can allow player 1 to access the resources of players 2, 3 and 4, in situation where the link between 1 and 2 fails. By contrast, under node failure, this additional link can never allow player 1 to get access to the resources of players 2 and 4 when player 2 fails; it can only allow access to the resources of player 3. Moreover, under node failure, in Fig. 1, player 1's payoff will not change if players 3 and 4 add a link between themselves. However, this link will improve her expected payoff under link failure. Finally, under node failure, player 1 also needs to take into account her own survival probability while computing her payoffs before adding a costly link. Thus, ensuring connectivity in the two different models may require different strategies.

Our focus in this note is on connectivity in the network. To study this we introduce the notion of k-Node Superconnectivity which checks whether a network is still connected after the deletion of any k nodes. Using this definition, we then identify sufficient conditions for both Nash and efficient networks.

2. Preliminaries

Graph-theoretic concepts. A (simple directed) **network** $\boldsymbol{g}$ is a pair of sets (N, E) where N is a set of nodes and $E \subset N \times N$ is a set of links with $(i, i) \notin E$ for all $i \in N$. We denote by $(i, j) \in E$ the link from i to j. Let G be the set of all (simple directed) networks whose set of vertices is N.

A **chain** in $\boldsymbol{g}$ between node j and node $i \neq j$, is an alternating sequence of distinct nodes $i_0, i_1, \ldots, i_m$ such that $i_0 = i, i_m = j$, and an alternating sequence of distinct links such that for $k = 0, \ldots, m-1$, $(i_k, i_{k+1}) \in E$ or $(i_{k+1}, i_k) \in E$. A network $\boldsymbol{g}$ is **connected** if there is a chain in $\boldsymbol{g}$ between all nodes $i, j \in N$. A subnetwork $\boldsymbol{g}^{N'} = (N', E')$ induced by $N' \subseteq N$ consists of a set of nodes N' and a set of links $E' \subset N' \times N'$ such that $(i, j) \in E'$ if and only if $(i, j) \in E$ for every pair $(i, j) \in N' \times N'$. Let $S(\boldsymbol{g})$ be the set of all subnetworks of $\boldsymbol{g}$ induced by all subsets $N' \subseteq N$. Note that for each subset $N' \subseteq N$ there is a unique subnetwork of $\boldsymbol{g}$ which belongs to $S(\boldsymbol{g})$. A **component** $\boldsymbol{g}^{N'}$ of $\boldsymbol{g}$ is a connected (induced) subnetwork of $\boldsymbol{g}$ such that for all $N'' \subseteq N$ with $N'' \supset N'$, $\boldsymbol{g}^{N''}$ is not connected. Finally, a network $\boldsymbol{g} \in G$ is **essential** if $(i, j) \in E$ implies $(j, i) \notin E$.

We now present two definitions that will play an important role in our analysis. A set of nodes $N' \subseteq N$ in a connected network $\boldsymbol{g}$ is **critical** if $\boldsymbol{g}^{N \setminus N'}$ is not connected. A network is k-**Node Super Connected** (k-NSC) if no set of k nodes or less is critical. To avoid triviality, we set $k < n - 2$.

Players and strategies. The set of players is identified with the set of nodes $N = \{1, \ldots, n\}$, $n \geq 3$. For each player $i \in N$, a pure strategy is a vector $\boldsymbol{g}_i = (g_{i,1}, \ldots, g_{i,i-1}, 0, g_{i,i+1}, \ldots, g_{i,n}) \in \{0, 1\}^n$. Here $g_{i,j} = 1$ means that player i forms a link with player j, whereas $g_{i,j} = 0$ means that i does not form this link. Let $\boldsymbol{g}_{-i} = (\boldsymbol{g}_1, \ldots, \boldsymbol{g}_{i-1}, \boldsymbol{g}_{i+1}, \ldots, \boldsymbol{g}_n)$ be the profile of strategies of all players except i. We focus only on pure strategies. The set of all pure strategies of player i is denoted by $\mathcal{G}_i$, with $\mathcal{G}_i = \{0, 1\}^{N \setminus \{i\}}$. The joint strategy space is denoted by $\mathcal{G} = \mathcal{G}_1 \times \cdots \times \mathcal{G}_n$. Note that there is a one-to-one correspondence between $\mathcal{G}$ and G the set of simple directed networks with vertex set N. Hence with a slight abuse of notation, we identify the strategy profile $(\boldsymbol{g}_1, \ldots, \boldsymbol{g}_n) \in \mathcal{G}$ with the network $\boldsymbol{g} = (N, E)$ where $g_{i,j} = 1$ if and only if $(i, j) \in E$.

Payoff. Player i incurs a cost $c > 0$ for each link she forms. We consider the **two-way flow of information** model, where both the agents involved in a link can access the resources (or information) of the other agent regardless of which agent initiates the link. Moreover, player i obtains resources from player j if there exists a chain between i and j. We denote by $N_i(\boldsymbol{g}) = \{j \in N : j \neq i,$ there exists a chain in $\boldsymbol{g}$ between i and $j\}$ the set of players whom i can access or "observe" in network $\boldsymbol{g}$.

In our context, players may stay put (i.e. node failure occurs) or appear (i.e. node failure does not occur). It follows that the network formed by the players can be different from the actual network observed. Hence we introduce the notion of realization to capture the effects of this assumption. Formally, a realization of $\boldsymbol{g}$, $\boldsymbol{g}^{N'} \in S(\boldsymbol{g})$, is a subnetwork of $\boldsymbol{g}$ where all players in N' are functioning and all players in $N \setminus N'$ are not functioning. Following the strategic reliability literature, assume the probability of node failure to be identical and independent, where the survival probability of every node is given by $p \in (0, 1)$. Given $\boldsymbol{g}$, the probability of subnetwork $\boldsymbol{g}^{N'}$ being realized is:

$$\lambda(\boldsymbol{g}^{N'}) = p^{|N'|}(1-p)^{n-|N'|}.$$

Note that for $\boldsymbol{g}, \boldsymbol{h} \in G$ we have $\lambda(\boldsymbol{g}^{N'}) = \lambda(\boldsymbol{h}^{N'})$ for all $N' \in 2^N$. This property is important for establishing Proposition 2.

We now define the function $B_i(\boldsymbol{g})$ as the expected benefit of player i in network $\boldsymbol{g}$. Summing over all possible realizations of the network we get:

$$B_i(\boldsymbol{g}) = V \sum_{N' \in 2^N} \lambda(\boldsymbol{g}^{N'}) |N_i(\boldsymbol{g}')|, \tag{1}$$

where V is the value of information that i gets from an agent with whom he is connected to directly or indirectly. Wlog we set $V = 1$. Using Eq. (1) we define i's expected payoff, that takes into account both costs and benefits as:

$$u_i(\boldsymbol{g}) = B_i(\boldsymbol{g}) - c \sum_{j \in N} g_{i,j}. \tag{2}$$

Nash networks. With a slight abuse of notation, we identify the pair $(\boldsymbol{g}_i, \boldsymbol{g}_{-i})$ with the network $\boldsymbol{g}$. A strategy $\boldsymbol{g}_i$ is a **best response** of player i to $\boldsymbol{g}_{-i}$ if

$$u_i(\boldsymbol{g}_i, \boldsymbol{g}_{-i}) \geq u_i(\boldsymbol{g}'_i, \boldsymbol{g}_{-i}), \quad \text{for all } \boldsymbol{g}'_i \in \mathcal{G}_i.$$