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Equivalence of efficiency notions for ordinal assignment problems[☆]

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HIGHLIGHTS

- We study four efficiency notions in ordinal assignment problems.
- We show that *sd*-, *dl*-, and *ul*-efficiency are equivalent.
- We provide conditions for equivalence of the three notions and ex post efficiency.
- Our conditions are sufficient and necessary for the equivalence.

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ABSTRACT

In ordinal (probabilistic) assignment problems, each agent reports his preference rankings over objects and receives a lottery defined over those objects. A common efficiency notion, *sd*-efficiency, is obtained by extending the preference rankings to preferences over lotteries by means of (first-order) stochastic dominance. Two alternative efficiency notions, which we call *dl*- and *ul*-efficiency, are based on downward and upward lexicographic dominance, respectively. We show that *sd*-, *dl*-, and *ul*-efficiency are all equivalent. Noting that the three efficiency notions are a refinement of ex post efficiency—another well-known efficiency notion—we also identify sufficient and necessary conditions on preference profiles under which ex post efficiency is equivalent to the three notions.

1. Introduction

In ordinal (probabilistic) assignment problems, agents submit ordinal preferences over indivisible commodities, called objects, and receive lotteries defined over those objects. Ordinal assignment problems abound in reality. When public schools allocate seats, when colleges assign dormitory rooms, or when local governments allot public housing units, those participating in a central distribution mechanism only report their preference rankings. In such mechanism design environments, a main objective is to ensure that an efficient allocation be chosen. In this paper, we focus on four notions of efficiency in the ordinal assignment literature and explore their equivalence. First, we show that three of these notions are equivalent. Further, observing that the three notions are a refinement of the fourth notion, we identify sufficient and necessary conditions on preference profiles under which the four notions are equivalent.

A common efficiency notion is obtained by extending preference rankings to preferences over lotteries by means of (first-order) stochastic dominance, which we call the *sd*-extension ("*sd*" stands for stochastic dominance). An assignment is *sd*-efficient if it cannot be Pareto improved with respect to the preferences so obtained, that is, it is not stochastically dominated by any other

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assignment (Bogomolnaia and Moulin, 2001).[1] The *sd*-extension gives incomplete preferences (some lotteries are not comparable). In contrast with this, two other extensions give complete preferences and give rise to different efficiency notions.

The first notion uses the downward lexicographic extension, or the *dl*-extension. The preferences obtained by the *dl*-extension compare probabilities of lotteries lexicographically, starting from the most preferred object. That is, given two lotteries, the lottery assigning a higher probability to the most preferred object is preferred; if the two lotteries assign the same probability, the lottery assigning a higher probability to the second most preferred object is preferred; and so on. Thus, the preferences obtained by the *dl*-extension lexicographically maximizes probabilities for preferred objects. Applying the standard Pareto dominance criterion to such lexicographic preferences yields *dl*-efficiency.

The other efficiency notion uses the upward lexicographic extension, or the *ul*-extension. The *ul*-extension implements the spirit of lexicographic comparison of lotteries in the opposite way: a lottery that lexicographically minimizes probabilities for less preferred objects is preferred. Combined with the Pareto dominance criterion, the *ul*-extension gives rise to *ul*-efficiency.

The *dl*- and *ul*-extensions respect stochastic dominance: if a lottery stochastically dominates another, the preferences obtained by the *dl*- or *ul*-extension prefers the former to the latter. Moreover, both extensions give complete preferences. These observations signal the possibility that even when it is impossible to Pareto improve an assignment in the *sd*-extension sense, it may be possible to Pareto improve it in the *dl*- or *ul*-extension sense. Yet we show that this is not the case; that is, *sd*-, *dl*-, and *ul*-efficiency are all equivalent (Theorem 1).

Given the equivalence of *sd*-, *dl*-, and *ul*-efficiency, we next analyze their logical relation with another well-known efficiency notion, ex post efficiency. Ex post efficiency requires that an assignment be a lottery over efficient deterministic assignments. An ex post efficient assignment may not be *sd*-efficient (Bogomolnaia and Moulin, 2001). Even worse, ex post efficiency fails to exclude even "wasteful" assignments, namely those that do not fully allocate probability shares of a desired object (Erdil, 2014). However, the two efficiency notions are equivalent for some preference profiles. Motivated by this, we provide two conditions on preference profiles that are jointly sufficient and necessary for the equivalence of ex post efficiency and *sd*-efficiency (Theorem 2).

The equivalence of ex post efficiency and *sd*-efficiency has implications for a widely used mechanism, the random priority mechanism (see Section 2 for a definition). The random priority mechanism is ex post efficient, but it may choose an *sd*-inefficient assignment and sometimes even a wasteful one (Erdil, 2014). It turns out that if the random priority mechanism is wasteful or *sd*-inefficient for some problem, it is precisely because ex post efficiency fails to imply non-wastefulness or *sd*-efficiency for that problem (Propositions 2 and 3).

Several recent papers take the ordinal approach based on the *dl*-extension and study efficiency, incentive, and fairness properties of mechanisms in various contexts. See Bogomolnaia (2015), Schulman and Vazirani (2012), Saban and Sethuraman (2014), Aziz et al. (2015) and Alcalde and Silva-Reus (2013) for object assignment; Alcalde (2013) for house allocation with existing tenants; and Aziz et al. (2014) for Arrovian voting. Cho (2016) introduces a unified setup to study the role of extensions in a general mechanism design environment. He focuses on incentive properties for ordinal mechanisms. Here our focus is on efficiency.

[1] Bogomolnaia and Moulin (2001) call this notion "ordinal efficiency". Since the notion is based on stochastic dominance and since other efficiency notions we propose are also ordinal in a sense, we use the term "sd-efficiency", which is due to Thomson (2011).

2. The model and efficiency notions

We consider the problem of allocating objects to agents using lotteries. Let $\boldsymbol{N}$ be a finite set of agents and $\boldsymbol{A}$ a finite set of objects. Let $\boldsymbol{n} \equiv |N| \geq 2$ and $\boldsymbol{m} \equiv |A| \geq 2$. Denote agents by i, j, i', j' and objects by a, b, a', b'. For each $a \in A$, $\boldsymbol{q_a} \in \mathbb{N}$ copies of object a are available. We assume that $\sum_{a\in A} q_a \geq n$, so that it is possible to allocate an object to each agent. Object a is a **null object** if $q_a \geq n$.[2]

Each agent $i \in N$ has a linear (i.e., complete, transitive, and anti-symmetric) preference relation $\boldsymbol{R_i}$ over A. Let $\boldsymbol{\mathcal{R}(A)}$ be the set of all such preference relations. Let $\boldsymbol{P_i}$ and $\boldsymbol{I_i}$ be the strict preference and indifference relations associated with R_i. With $(q_a)_{a\in A}$ fixed, an (assignment) **problem** is a preference profile $\boldsymbol{R} \equiv (R_i)_{i\in N}$. Let $\boldsymbol{\mathcal{R}(A)^N}$ be the set of all problems.

A **lottery** over A is a probability distribution over A. Let $\boldsymbol{\Delta A}$ be the set of all lotteries over A. A (probabilistic) **assignment** is a profile $\boldsymbol{x} \equiv (x_i)_{i\in N}$ such that (i) for each $i \in N$, $x_i \in \Delta A$; and (ii) for each $a \in A$, $\sum_{i\in N} x_{ia} \leq q_a$ (x_{ia} is the probability of agent i receiving object a). Let $\boldsymbol{X}$ be the set of all assignments. An assignment can be viewed as an $n \times m$ matrix whose rows are indexed by agents and columns by objects. An assignment is **deterministic** if for each $i \in N$, x_i is a degenerate lottery (i.e., for some $a \in A$, $x_{ia} = 1$). Let $\boldsymbol{D}$ be the set of all deterministic assignments. Each probabilistic assignment can be represented as a convex combination of deterministic assignments (Birkhoff, 1946; von Neumann, 1953; Budish et al., 2013). If assignment x is deterministic, then identifying objects with degenerate lotteries, we sometimes write, e.g., $x_i \, P_i \, b$.

An (assignment) **mechanism** $\varphi : \mathcal{R}(A)^N \to X$ associates with each problem an assignment. Below are some examples of mechanisms. Given a linear priority order $\prec$ over N, the **sequential priority mechanism associated with** $\prec$, denoted $\boldsymbol{SP^{\prec}}$, allocates objects according to the priority order $\prec$: first, the agent with the highest priority according to $\prec$ is assigned his most preferred object in A; then the agent with the second highest priority according to $\prec$ is assigned his most preferred object among those still available; and so on. By definition, the sequential priority mechanisms are deterministic. The **random priority mechanism**, denoted ***RP***, is the simple average of all $n!$ sequential priority mechanisms.

Let $R \in \mathcal{R}(A)^N$ be a problem. A deterministic assignment $x \in D$ is **deterministically efficient for** $\boldsymbol{R}$ if there is no $y \in D$ such that (i) for each $i \in N$, $y_i \, R_i \, x_i$; and (ii) for some $j \in N$, $y_j \, P_j \, x_j$. An assignment $x \in X$ is **ex post efficient for** $\boldsymbol{R}$ if it can be written as a convex combination of deterministic assignments, each of which is deterministically efficient for R. Let $\boldsymbol{E^{xp}(R)}$ be the set of ex post efficient assignments for R.

Ex post efficiency is defined with no reference to agents' preferences over lotteries. However, one can extend preferences over objects to preferences over lotteries and then define alternative notions of efficiency with respect to the preferences so obtained. Bogomolnaia and Moulin (2001) study the extension procedure based on (first-order) stochastic dominance and the associated efficiency notion. Together with the latter extension, we consider two other extensions based on lexicographic dominance.

To present these ideas formally, we introduce a concept from Cho (2016). Let $\boldsymbol{\mathcal{R}(\Delta A)}$ be the set of all preferences over lotteries over A. An **extension** is a mapping $e : \mathcal{R}(A) \to \mathcal{R}(\Delta A)$ such that for each $R_i \in \mathcal{R}(A)$, the restriction of $e(R_i)$ to A coincides with R_i. For each $R_i \in \mathcal{R}(A)$, let $\boldsymbol{R_i^e} \equiv e(R_i)$. The strict preference and indifference relations associated with R_i^e are denoted by $\boldsymbol{P_i^e}$ and $\boldsymbol{I_i^e}$, respectively. The stochastic dominance extension, or the ***sd*-extension**, is defined as follows: for each $R_i \in \mathcal{R}(A)$ and each pair

[2] This definition permits the possibility of several null objects.